Games and Economic Behavior 140 (2023) 316–340

Contents lists available at ScienceDirect

Games and Economic Behavior

journal homepage: www.elsevier.com/locate/geb

Simultaneous 2nd price item auctions with no-underbidding ☆

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ARTICLE INFO

Article history:
Received 25 January 2021
Available online 28 March 2023

Keywords:
Algorithmic game theory
Simultaneous item bidding auctions
Price of anarchy

ABSTRACT

The literature on the Price of Anarchy (PoA) of simple auctions employs a no-overbidding assumption but has completely overlooked the no-underbidding phenomenon, which is evident in empirical studies on variants of the second price auction. In this work, we provide a theoretical foundation for the no-underbidding phenomenon. We study the PoA of simultaneous 2nd price auctions (S2PA) under a new natural condition of *no underbidding*, meaning that agents never bid on items less than their marginal values. We establish improved (mostly tight) bounds on the PoA of S2PA under no-underbidding for different valuation classes, in both full-information and incomplete information settings. Specifically, we show that the PoA is at least 1/2 for general monotone valuations, which extends to Bayesian PoA with arbitrary correlated distributions. We also establish a tight PoA bound of 2/3 for S2PA with XOS valuations, under no-overbidding and no-underbidding, which extends to Bayesian PoA with independent distributions.

1. Introduction

Simple auctions are often preferred in practice over complex truthful auctions. Starting with the seminal paper of Christodoulou et al. (2008), a lot of effort has been given to the study of *simultaneous item auctions*.

In simultaneous item auctions with n bidders and m items, every bidder i has a valuation function $v_i : 2^{[m]} \to \mathbb{R}^+$, where $v_i(S)$ is the value bidder i assigns to set $S \subseteq [m]$. Despite the combinatorial structure of the valuation, bidders submit bids on every item *separately and simultaneously*. In simultaneous first-price auctions (S1PA) every item is sold in a 1st-price auction; i.e., the highest bidder wins and pays her bid, whereas in simultaneous second-price auctions (S2PA) every item is sold in a 2nd-price auction; i.e., the highest bidder wins and pays the 2nd highest bid.

Clearly, these auctions are not truthful; Indeed, bidders don't even have the language to express their true valuations, as they are restricted to place bids on individual items, while their valuations apply to bundles of items, possibly in a non-linear way. Furthermore, even in scenarios where bidders can express their valuations by individual bids, e.g., unit-demand valuations, revealing one's true value is not a dominant strategy. Consider, e.g., Example 1.1, and suppose bidder 1 places bids $1/2, 1$ on items x, y, respectively. Then, bidder 2 is better off placing a single non-zero bid rather than placing her true values on the items. The performance of these auctions is often quantified by the *price of anarchy* (PoA), which measures their performance in equilibrium. Specifically, the PoA is defined as the ratio between the performance of an auction in its worst equilibrium and the performance of the optimal outcome. The price of anarchy in auctions has been of great interest

☆ This work was partially supported by the Israel Science Foundation (grant number 1219/09), and by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No. 866132).

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https://doi.org/10.1016/j.geb.2023.03.009
0899-8256/

Table 1
Previous results for PoA of S2PA under the NOB assumption. PoA is the price of anarchy under full information; iBPoA and BPoA are the Bayesian PoA under independent valuation distributions and correlated valuation distributions, respectively. † This result assumes *strong* no-overbidding, i.e. the sum of player bids on *any* set of items never exceeds its value for that set. All results are tight, except those marked with ∼. Results derived as a special case of a more general result (to their right) are marked with ←.

		UD / SM		XOS		SA		MON	
NOB	PoA	$\frac{1}{2}$	←	$\frac{1}{2}$	Christodoulou et al. (2016a)	$\frac{1}{2}^{\dagger}$	Bhawalkar and Roughgarden (2011)	$O\left(\frac{1}{\sqrt{m}}\right)^{\sim}$	Hassidim et al. (2011); Feldman et al. (2013)
	iBPoA	$\frac{1}{2}$	←	$\frac{1}{2}$	Christodoulou et al. (2016a)	$\frac{1}{4}^{\sim}$	Feldman et al. (2013)	$O\left(\frac{1}{\sqrt{m}}\right)^{\sim}$	Hassidim et al. (2011); Feldman et al. (2013)
	BPoA	$O\left(\frac{1}{n^{1/4}}\right)^{\sim}$	Bhawalkar and Roughgarden (2011)	$O\left(\frac{1}{n^{1/4}}\right)^{\sim}$	Bhawalkar and Roughgarden (2011)	$O\left(\frac{1}{n^{1/4}}\right)^{\sim}$	Bhawalkar and Roughgarden (2011)	$O\left(\frac{1}{n^{1/4}}\right)^{\sim}$	Bhawalkar and Roughgarden (2011)

to the AI community, see the influential survey of Roughgarden et al. (2017), as well as recent work on learning dynamics in multi-unit auctions and games Foster et al. (2016); Brânzei and Filos-Ratsikas (2019).

PoA and BPoA of S2PA: Background. The price of anarchy has been studied both in complete and incomplete information settings. In the former case, all valuations are known by all bidders. In the latter case, every bidder knows her own value and the probability distribution from which other bidder valuations are drawn. The common equilibrium notion in this case is *Bayes Nash equilibrium*, and the performance is quantified by the *Bayesian PoA* (BPoA) measure.

There are pathological examples showing that the PoA of S2PA can be arbitrarily bad, even in the simplest scenario of a single item auction with two bidders Christodoulou et al. (2016a). A common approach towards overcoming such pathological examples is the *no overbidding* (NOB) assumption, stating that the sum of player bids on the set of items she wins never exceeds its value. Consequently, all PoA results of S2PA use the NOB assumption.

The PoA and BPoA of simultaneous item auctions depend on the structure of the valuation functions. An important class is that of subadditive valuations, also known as *complement-free* valuations, where $v(S) + v(T) \geq v(S \cup T)$ for every sets of items S, T. A hierarchy of complement-free valuations is given in Lehmann et al. (2006), including unit-demand, submodular, XOS, and subadditive valuations, with the following strict containment relation: $unit - demand \subset submodular \subset XOS \subset subadditive$ (see Section 2.2 for formal definitions). Clearly, the PoA can only degrade as one moves to a larger valuation class. PoA and BPoA results under the no-overbidding assumption for the different classes have been obtained by Christodoulou et al. (2008) and follow-up work Roughgarden (2009); Bhawalkar and Roughgarden (2011); Hassidim et al. (2011); Roughgarden (2012); Feldman et al. (2013); Syrgkanis and Tardos (2013); Christodoulou et al. (2016b), and are summarized in Table 1. MON refers to the class of all monotone valuations.

1.1. No underbidding (NUB)

Let $\mathbf{v} = (v_1, \ldots, v_n)$ be a valuation profile, where v_i is the valuation of bidder i. Also, let $b_i = (b_{i1}, \ldots, b_{im})$ denote the bid vector of bidder i, where b_{ij} is the bid of bidder i for item j, and let $\mathbf{b} = (b_1, \ldots, b_n)$ be the bid profile of all bidders. Consider the following example (taken from Christodoulou et al. (2016a)), showing that the PoA for unit-demand valuations is at most 1/2 (a valuation v is unit-demand if there exist $v(1), \ldots, v(m)$, such that $v(S) = \max_{j \in S} v(j)$ for every set of items S).

Example 1.1. 2 bidders and 2 items, x, y. Bidder 1 is unit-demand with values $v_1(x) = 2$, $v_1(y) = 1$. Bidder 2 is unit-demand with values $v_2(x) = 1$, $v_2(y) = 2$. Consider the following bid profile, which is a pure Nash equilibrium (PNE) that adheres to NOB: $b_{1x} = b_{2y} = 0$, and $b_{1y} = b_{2x} = 1$. Under this bid profile, bidders 1 and 2 receive items y and x, respectively, for a social welfare of 2. The optimal welfare is 4.

Let us take a closer look at the Nash equilibrium in Example 1.1. In this equilibrium bidder 1 prefers item x, yet bids 0 on item x, and gets item y instead. Bidder 1's marginal value for item x, given her current allocation (item y), is $v_1(x \mid y) = v_1(xy) - v_1(y) = 1$. Given her current allocation y, bidding 0 on item x is *weakly dominated* by bidding 1 on x. Indeed, if bidder 1 receives item x, in addition to item y, her additional value is 1 and she pays at most 1.

If a bidder bids on an item less than the item's marginal value, we say that she *underbids* (see Definition 4.1). In Example 1.1, bidder 1 underbids on item x. In Section 4 we show that underbidding in a 2nd price auction is *weakly dominated* in some precise technical sense.

While a Nash equilibrium is a descriptive, static notion, it is based on the underlying assumption that players engage in some dynamics, where they keep best responding to the current situation until a stable outcome is reached. We claim that in this dynamics, it is not likely that a player would bid on an item less than its marginal value. This is exactly what the no-underbidding assumption captures.

No-underbidding is not only a mere theoretical exercise. In second price auctions a lot of empirical evidence suggests that bidders tend to overbid, but not underbid Kagel and Levin (1993); Harstad (2000); Cooper and Fang (2008); Roider and Schmitz (2012). It seems that "laboratory second-price auctions exhibit substantial and persistent overbidding, even with prior experience" Harstad (2000). Indeed, a typical bluffing behavior relies on aggressive overbidding, where one bids unreasonably high to cause others pull out of the auction. The no-underbidding assumption is also consistent with the assumption made by Nisan et al. (2011) that bidders break ties in favor of the highest bid that does not exceed their value. Yet, the POA literature employs no-overbidding as a standard assumption, and overlooked the no-underbidding phenomenon. The objective of this work is to better tie the theoretical work in this area to empirical evidence, by providing a theoretical foundation for the no-underbidding phenomenon. We view this study as a first step toward the study of equilibria satisfying no underbidding; we hope that this study will lead to empirical work that further validates and refines the proposed no-underbidding notions.

Intuitively, no underbidding can improve welfare performance, as it drives item prices up, so that items become less attractive to low-value players. Consequently, bad equilibria, in which items are allocated to players with relatively low value, are excluded. A natural question is:

Main Question. *What is the performance (measured by PoA/BPoA) of simultaneous 2nd price item auctions under no underbidding?*

1.2. Our contribution

We first introduce the notion of *item no underbidding* (iNUB), where no agent underbids on items (see Definition 4.3). One might think that by imposing both NOB and iNUB, the optimal welfare will be achieved. This is indeed the case for a single item auction (where the optimal welfare is achieved by imposing any one of these assumptions alone). However, even a simple scenario with 2 items and 2 unit-demand bidders can have a PNE with sub-optimal welfare. This is demonstrated in the following example.

Example 1.2. 2 bidders, and 2 items: x, y. Bidder 1 is unit-demand with values $v_1(x) = 3, v_1(y) = 2$. Bidder 2 is unit-demand with values $v_2(x) = 2, v_2(y) = 3$. Consider the following PNE bid profile, which adheres to both NOB and iNUB: $b_{1x} = b_{2y} = 1, b_{1y} = b_{2x} = 2$. Under this bid profile, bidders 1 and 2 receive items y and x, respectively, for a social welfare of 4. The optimal welfare is 6. Thus, the PoA is 2/3.

To overcome existence issues, we state our results with respect to the *weak iNUB* notion (see Definition 4.5). Weak iNUB is a slightly weaker notion of item no-underbidding, which exists more generally and yet, provides the same price of anarchy bounds.

Submodular Valuations. Our first result states that 2/3 is the worst possible ratio for submodular valuations and bid profiles satisfying both NOB and weak iNUB, even in settings with incomplete information (with a product distribution over valuations), see Corollary 5.4.

Theorem [submodular valuations, weak iNUB and NOB]: For every market with submodular valuations,

- The PoA (even with respect to *coarse correlated equilibrium* (CCE)[1]) and the BPoA (for product or correlated distribution) of S2PA under weak iNUB are both at least $\frac{1}{2}$ (see Corollary 5.1).
- The PoA (even with respect to CCE), and the BPoA (for product distribution) of S2PA under NOB and weak iNUB are both at least $\frac{2}{3}$ (see Corollary 5.4).

The above results are tight, even with respect to PNE, even for unit-demand valuations, and even for iNUB.

Moreover, the last theorem extends to α-submodular valuations, defined as $v(j \mid S) \geq \alpha \cdot v(j \mid T)$ for every $S \subseteq T$. We show that the (B)PoA degrades gracefully with the parameter α; namely the PoA with respect to CCE and the BPoA are at least $\frac{\alpha}{1+\alpha}$ under weak iNUB and at least $\frac{2\alpha}{2+\alpha}$ under NOB and weak iNUB (see Corollaries 5.1 and 5.4, respectively).

Beyond Submodular Valuations. The above bounds do not carry over beyond (α-)submodular valuations. Consider first XOS valuations, defined as maximum over additive valuations. We show that the (B)PoA of S2PA with XOS valuations under iNUB is $\theta(\frac{1}{m})$ (the lower bound is given in Appendix A.1, and the upper bound is given in Example 6.1). Moreover, for XOS valuations, iNUB may not provide any improvement over NOB alone. In particular, in Example A.1 the PoA with NOB and iNUB is 1/2, matching the guarantee provided by NOB alone.

To the best of our knowledge, this is the first PoA separation between submodular and XOS valuations in simultaneous item auctions. In fact, the PoA of simultaneous item auctions is often the same for the entire range between unit-demand and XOS. This separation suggests that XOS is "far" from submodular. Indeed, in Appendix B.1 we show that XOS is not

[1] The set of coarse correlated equilibria (CCE) is a superset of Nash equilibria; a formal definition appears in Section 2.4.

Table 2
New results for S2PA price of anarchy lower bound. PoA is the price of anarchy under full information, iBPoA is the Bayesian PoA under independent valuation distributions, and BPoA is the Bayesian PoA under correlated valuation distributions. All results are tight and apply also for the *weak* versions of iNUB and sNUB. Results derived as a special case of a more general result (to their right) are marked with ←.

		UD / SM		XOS		SA		MON
iNUB	(B)PoA	$\frac{1}{2}$		$\Theta\left(\frac{1}{m}\right)$				arbitrarily bad
sNUB	(B)PoA	$\frac{1}{2}$	←	$\frac{1}{2}$	←	$\frac{1}{2}$	←	$\frac{1}{2}$
NOB+	PoA	$\frac{2}{3}$	←	$\frac{2}{3}$		$\frac{2}{3}$		$\frac{1}{2}$
sNUB	iBPoA	$\frac{2}{3}$	←	$\frac{2}{3}$		$\frac{1}{2}$	←	$\frac{1}{2}$
	BPoA	$\frac{1}{2}$	←	$\frac{1}{2}$	←	$\frac{1}{2}$	←	$\frac{1}{2}$

α-submodular for any fixed $0 < \alpha \leq 1$, even in settings with identical items. Moreover, beyond subadditive valuations, the PoA can be arbitrarily bad under bid profiles satisfying iNUB (see Appendix A.2).

To deal with valuations beyond submodular, we consider a different no underbidding assumption, which applies to *sets of items*. For two sets S, T, the *marginal value* of T given S is defined as $v(T \mid S) = v(S \cup T) - v(S)$. A bidder is said to not underbid on a set of items S under bid profile $\mathbf{b}$ if $\sum_{j \in S} b_{ij} \geq v_i(S \mid S_i(\mathbf{b}))$, where $S_i(\mathbf{b})$ is the set of items won by player i under bid profile $\mathbf{b}$. The new condition, *set no underbidding* (sNUB), imposes the set no underbidding condition on every bidder i and every set $T \subseteq [m] \backslash S_i(\mathbf{b})$ (see Definition 4.4). As before, to deal with existence issues, we state our results with respect to a weaker notion of sNUB, called *weak sNUB* (see Definition 4.6), which exists more generally and provides the same price of anarchy guarantees as sNUB.

With the *set* no-underbidding definition (either sNUB or weak sNUB), the 2/3 PoA extends to subadditive valuations in full information settings, and to XOS valuations even in incomplete information settings (with product distributions), see Corollary 6.1. We view this as one of the main results in the paper.

Theorem [subadditive and XOS valuations, NOB and weak sNUB]: For every market with subadditive valuations, the PoA with respect to CCE of S2PA under strong NOB and weak sNUB is at least 2/3 (see Theorem 7.2). For every market with XOS valuations, the BPoA (under product distribution) of S2PA under NOB and weak sNUB is at least 2/3 (see Corollary 6.1). Both results are tight, even for sNUB.

Our second main result is the following. For incomplete information we show that the BPoA of subadditive valuations is at least 1/2 for *any* joint distribution (even correlated) and it can be obtained in a much stronger sense, namely for every bid profile with non-negative sum of utilities (even a non-equilibrium profile) satisfying weak sNUB. This also holds for markets with arbitrary monotone valuations.

Theorem [Arbitrary valuations, weak sNUB]: For every market (arbitrary monotone valuations), the PoA with respect to CCE and the BPoA (for any joint distribution) of S2PA under weak sNUB is at least 1/2 (see Corollary 4.1).

The above results are summarized in Table 2.

Equilibrium existence. PoA results make sense only when the corresponding equilibrium exists. We show that for unit-demand valuation there always exists a PNE bid profile which satisfies both NOB and sNUB. We also show that for submodular valuations there always exists a PNE bid profile which satisfies sNUB. Furthermore, every market with XOS valuations admits a PNE satisfying weak sNUB and NOB. For subadditive valuations, a PNE satisfying NOB might not exist. However, under a finite discretized version of the auction, a mixed Bayes Nash equilibrium is guaranteed to exist, and we show that there is at least one bid profile that admits both weak sNUB and NOB with arbitrary monotone valuation functions. Table 3 summarizes the PNE existence and non-existence results.

S1PA vs. S2PA. Interestingly, our results shed new light on the comparison between simultaneous 1st and 2nd price auctions. Table 4 specifies BPoA lower bounds for S1PA and S2PA under NOB, assuming independent valuation distributions. According to these results, one may conclude that S1PA perform better than their S2PA counterparts.

Our new results shed more light on the relative performance of S2PA and S1PA. When considering both no overbidding and no underbidding, the situation flips, and S2PA are superior to S1PA.[2] For XOS valuations, the $1 - 1/e$ bound for S1PA persists, but for S2PA the bound improves from $\frac{1}{2}(< 1 - \frac{1}{e})$ to $\frac{2}{3}(> 1 - \frac{1}{e})$. For subadditive valuations and independent valuation distributions, S2PA under weak sNUB performs as well as S1PA (achieving BPoA of $\frac{1}{2}$), however in S2PA the $\frac{1}{2}$ bound holds also for correlated valuation distributions. For valuations beyond subadditive, S2PA performs better ($\frac{1}{2}$ for S2PA and less than $\frac{1}{2}$ for S1PA).

[2] Note that the no overbidding assumption comes for free in 1st price auctions, and the no underbidding assumption is not reasonable in 1st price auctions, where bidders may definitely wish to underbid on items, as they pay their bids.

Table 3
S2PA PNE existence and non-existence results for the various valuation classes and bid profile types. Cases in which a S2PA PNE always exists are marked with "+", while cases in which there exists an instance that has no S2PA PNE are marked with "-". Results derived as a special case of a more general result (to their right) are marked with ←. Similarly, results derived from a more restricted case (to their left) are marked with →.

	UD	SM	XOS	SA	MON
NOB + sNUB	+ Proposition 5.4	- Proposition 5.5	- →	- →	- →
sNUB	+ ←	+ Proposition 5.6	- Proposition 6.1	- →	- →
NOB + weak sNUB	+ ←	+ ←	+ Theorem 6.1	- Fu et al. (2012)*	- →
weak sNUB	+ ←	+ ←	+ ←	+ ←	+ Proposition 7.1

* Fu et al. (2012) showed that S2PA PNE with NOB may not exist for subadditive valuations.

Table 4
Bayesian price of anarchy results for simultaneous first price and second price auctions. (corr) refers to bounds that hold also for correlated distributions. Results derived from the current paper are marked with *. Results derived as a special case of a more general result (to their right) are marked with ←. The results in the third row apply also with respect to weak sNUB.

		UD / SM		XOS		SA		MON	
S2PA	NOB	$\frac{1}{2}$	←	$\frac{1}{2}$	Christodoulou et al. (2016a)	$\frac{1}{4}$	Feldman et al. (2013)	$O\left(\frac{1}{\sqrt{m}}\right)$	Hassidim et al. (2011); Feldman et al. (2013)
S1PA		$1-\frac{1}{e}$	←	$1-\frac{1}{e}$	Syrgkanis and Tardos (2013); Christodoulou et al. (2016b)	$\frac{1}{2}$	Feldman et al. (2013); Christodoulou et al. (2016b)	$\frac{1}{m}$	Hassidim et al. (2011)
S2PA	sNUB+NOB	$\frac{2}{3}$	←	$\frac{2}{3}$*		$\frac{1}{2}$ (corr)	←	$\frac{1}{2}$* (corr)	

1.3. Our techniques

The standard technique for establishing performance guarantees for equilibria of simple auctions (i.e., PoA results) is the *smoothness* framework (see the survey in Roughgarden et al. (2017)). Smoothness is a parameterized notion; an auction is said to be (λ, μ)-smooth if for any valuation profile $\mathbf{v}$ and any bid profile $\mathbf{b}$ there exists a bid $b_i^*(\mathbf{v})$ for each player i, s.t. $\sum_{i\in[n]} u_i(b_i^*(\mathbf{v}), \mathbf{b}_{-i}, v_i) \geq \lambda OPT(\mathbf{v}) - \mu SW(\mathbf{b}, \mathbf{v})$, where u_i is the utility of player i, $SW(\mathbf{b}, \mathbf{v})$ is the social welfare under $\mathbf{b}$ and $OPT(\mathbf{v})$ is the social welfare of an optimal allocation. It is quite straightforward to show that if an auction is (λ, μ)-smooth, then its PoA with respect to PNE is at least $\frac{\lambda}{1+\mu}$.

The power of the smoothness framework is in its extendability. While a lower bound on the PoA with respect to PNE follows easily from the smoothness property, this lower bound extends to the PoA with respect to CCE and with respect to the Bayesian PoA in games with incomplete information Roughgarden (2009, 2012); Roughgarden et al. (2017); Syrgkanis and Tardos (2013).

Revenue Guaranteed Auctions. We introduce a new parameterized notion called *revenue guaranteed*. An auction is said to be (γ, δ)-revenue guaranteed if for every valuation profile $\mathbf{v}$ and bid profile $\mathbf{b}$ the revenue of the auction is bounded below by $\gamma \cdot OPT(\mathbf{v}) - \delta \cdot \mathbf{SW}(\mathbf{b}, \mathbf{v})$.

We show that in every (γ, δ)-revenue guaranteed auction, the social welfare in every bid profile with non-negative sum of utilities is at least a fraction $\frac{\gamma}{1+\delta}$ of the optimal welfare. Similarly to the smoothness framework, we augment our results with two extension theorems, one for PoA with respect to CCE, and one for BPoA in settings with incomplete information. Moreover, this result holds also in cases where the joint distribution of bidder valuations is correlated (whereas previous BPoA results hold only under a product distribution over valuations).

Combining the two tools of smoothness and revenue guaranteed, we get an improved bound. In particular, we show that in every auction that is both (λ, μ)-smooth and (γ, δ)-revenue guaranteed, the PoA with respect to CCE is at least $\frac{\lambda+\gamma}{1+\mu+\delta}$. The same holds for the BPoA under product valuation distributions.

Implications on Simultaneous Second Price Auctions. With this tool in hand, we analyze simultaneous 2nd price auctions with different valuation functions and different no underbidding conditions, where the goal is to establish revenue-guaranteed parameters that would imply PoA and BPoA bounds.

We first consider submodular and α-submodular valuations. We show that every S2PA with α-submodular valuations satisfying weak iNUB is (α, α)-revenue guaranteed. This directly gives a lower bound of $\frac{\alpha}{1+\alpha}$ on the BPoA of α-submodular valuations (and $1/2$ for submodular valuations). We also show that S2PA with α-submodular valuations satisfying NOB are $(\alpha, 1)$-smooth. Combining (α, α)-revenue guaranteed with $(\alpha, 1)$-smoothness gives a bound of $\frac{2\alpha}{2+\alpha}$ on the BPoA for every S2PA with α-submodular valuations with NOB and weak iNUB. For submodular valuations this gives the tight $2/3$ bound.

For valuations beyond α-submodular valuations, the iNUB condition is not helpful, so we turn to the stronger sNUB condition. We show that every S2PA with arbitrary monotone valuations satisfying weak sNUB is $(1, 1)$-revenue guaranteed for bid profiles with non-negative sum of utilities. This recovers the $1/2$ bound on PoA with respect to CCE and BPoA with correlated distributions for S2PA satisfying weak sNUB. For XOS valuations, we combine the last result with the known $(1, 1)$-smoothness to get the $2/3$ bound for S2PA satisfying NOB and weak sNUB (for PoA with respect to CCE and for BPoA with product distributions). For subadditive valuations, we apply the technique from Bhawalkar and Roughgarden (2011) to yield a tight bound of $2/3$ on the PoA with respect to CCE.

Note that our notion of revenue guarantee is unrelated to the revenue covering property introduced in Hartline et al. (2014). Their revenue covering property is defined for single-parameter settings, and relies heavily on some relationship between thresholds and revenue which does not apply in second price auctions.

2. Preliminaries

2.1. Auctions

Combinatorial auctions. In a combinatorial auction a set of m non-identical items are sold to a group of n players. Let $\mathcal{S}_i$ be the set of possible allocations to player i, $\mathcal{V}_i$ the set of possible valuations of player i, and $\mathcal{B}_i$ the set of actions available to player i. Similarly, we let $\mathcal{S} \subseteq \mathcal{S}_1 \times \ldots \times \mathcal{S}_n$ be the allocation space of all players, $\mathcal{V} = \mathcal{V}_1 \times \ldots \times \mathcal{V}_n$ be the valuation space, and $\mathcal{B} = \mathcal{B}_1 \times \ldots \times \mathcal{B}_n$ be the action space. An allocation function maps an action profile to an allocation $\mathbf{S} = (S_1, \ldots, S_n) \in \mathcal{S}$, where S_i is the set of items allocated to player i. A payment function maps an action profile to a non negative payment $\mathbf{P} = (P_1, \ldots, P_n) \in \mathbb{R}_+$, where P_i is the payment of player i. We assume that the valuation function $v_i : \mathcal{S}_i \to \mathbb{R}_+$ of a player i, where $v_i \in \mathcal{V}_i$, is monotone and normalized, i.e., $\forall S \subseteq T \subseteq [m]$, $v_i(S) \leq v_i(T)$ and also $v_i(\emptyset) = 0$. We let $\mathbf{v} = (v_1, \ldots, v_n)$ be the valuation profile. An outcome is a pair of allocation $\mathbf{S}$ and payment $\mathbf{P}$ and the revenue is the sum of all payments, i.e. $\mathcal{R}(\mathbf{b}) = \sum_{i\in[n]} P_i(\mathbf{b})$. We assume a quasi-linear utility function, i.e. $u_i(S_i, P_i, v_i) = v_i(S_i) - P_i$. We are interested in measuring the *social welfare*, which is the sum of bidder valuations, i.e., $SW(\mathbf{S}, \mathbf{v}) = \sum_{i\in[n]} v_i(S_i)$. Given a valuation profile $\mathbf{v}$, an optimal allocation is an allocation that maximizes the SW over all possible allocations. We denote by $OPT(\mathbf{v})$ the social welfare value of an optimal allocation.

Simultaneous item bidding auction. In a simultaneous item bidding auction (simultaneous item auction, in short) each item $j \in [m]$ is simultaneously sold in a separate auction. An action profile is a bid profile $\mathbf{b} = (b_1, \ldots, b_n)$, where $b_i = (b_{i1}, \ldots, b_{im})$ is an m-vector s.t. b_{ij} is the bid of player i for item j. The allocation of each item j is determined by the bids $(b_{1j}, \ldots, b_{nj})$. We use $S_i(\mathbf{b})$ to denote the items won by player i and $p_j(\mathbf{b})$ to denote the price paid for item j by the winner of item j. As allocation and payment are uniquely defined by the bid profile, we overload notation and write $u_i(\mathbf{b}, v_i)$ and $SW(\mathbf{b}, \mathbf{v})$.

In a simultaneous second price auction (S2PA), each item j is allocated to the highest bidder, who pays the second highest bid, i.e., $P_i = \sum_{j\in S_i(\mathbf{b})} \max_{k\neq i} b_{kj}$.

In a simultaneous first price auction (S1PA), each item j is allocated to the highest bidder, who pays her bid for that item, i.e., $P_i = \sum_{j\in S_i(\mathbf{b})} b_{ij}$.

Ties are broken arbitrarily but consistently.

Full information setting: solution concepts and PoA. In the full information setting, the valuation profile $\mathbf{v} = (v_1, \ldots, v_n)$ is known to all players. The standard equilibrium concepts in this setting are pure Nash equilibrium (PNE), mixed Nash equilibrium (MNE), correlated Nash equilibrium (CE) and coarse correlated Nash equilibrium (CCE), where $PNE \subset MNE \subset CE \subset CCE$. Following are the definitions of the equilibrium concepts. As standard, for a vector $\mathbf{y}$, we denote by $\mathbf{y}_{-i}$ the vector $\mathbf{y}$ with the ith component removed. Also, we denote with $\Delta(\Omega)$ the space of probability distributions over a finite set Ω.

Definition 2.1 *(Pure Nash Equilibrium (PNE)).* A bid profile $\mathbf{b} \in \mathcal{B}_1 \times \ldots \times \mathcal{B}_n$ is a PNE if for any $i \in [n]$ and for any $b'_i \in \mathcal{B}_i$, $u_i(\mathbf{b}, v_i) \geq u_i(b'_i, \mathbf{b}_{-i}, v_i)$.

Definition 2.2 *(Mixed Nash Equilibrium (MNE)).* A bid profile of randomized bids $\mathbf{b} \in \Delta(\mathcal{B}_1) \times \ldots \times \Delta(\mathcal{B}_n)$ is a MNE if for any $i \in [n]$ and for any $b'_i \in \mathcal{B}_i$, $\mathbb{E}_{\mathbf{b}}[u_i(\mathbf{b}, v_i)] \geq \mathbb{E}_{\mathbf{b}_{-i}}\left[u_i(b'_i, \mathbf{b}_{-i}, v_i)\right]$.

Definition 2.3 *(Correlated Nash Equilibrium (CE)).* A bid profile of randomized bids $\mathbf{b} \in \Delta(\mathcal{B}_1 \times \ldots \times \mathcal{B}_n)$ is a CE if for any $i \in [n]$ and for any mapping $b'_i(b_i)$, $\mathbb{E}_{\mathbf{b_{-i}}}[u_i(\mathbf{b}, v_i) \mid b_i] \geq \mathbb{E}_{\mathbf{b_{-i}}}\left[u_i(b'_i, \mathbf{b}_{-i}, v_i) \mid b_i\right]$.

Definition 2.4 *(Coarse Correlated Nash Equilibrium (CCE)).* A bid profile of randomized bids $\mathbf{b} \in \Delta(\mathcal{B}_1 \times \ldots \times \mathcal{B}_n)$ is a CCE if for any $i \in [n]$ and for any $b'_i \in \mathcal{B}_i$, $\mathbb{E}_{\mathbf{b}}[u_i(\mathbf{b}, v_i)] \geq \mathbb{E}_{\mathbf{b_{-i}}}\left[u_i(b'_i, \mathbf{b}_{-i}, v_i)\right]$.

For a given instance of valuations $\mathbf{v}$, the price of anarchy (PoA) with respect to an equilibrium notion E is defined as: $PoA(\mathbf{v}) = \inf_{\mathbf{b}\in E} \frac{\mathbb{E}_{\mathbf{b}}[SW(\mathbf{b},\mathbf{v})]}{OPT(\mathbf{v})}$. For example, the PoA with respect to PNE is $PoA(\mathbf{v}) = \inf_{\mathbf{b}\in PNE} \frac{SW(\mathbf{b},\mathbf{v})}{OPT(\mathbf{v})}$. The PoA for the other equilibrium types are defined in a similar manner. For a family of valuations V, $PoA(\mathrm{V}) = \min_{\mathbf{v}\in \mathrm{V}} PoA(\mathbf{v})$.

The following lemma will be useful in subsequent sections of this paper.

Lemma 2.1. *Consider an S2PA and a valuation* $\mathbf{v}$. *Let* $S^*(\mathbf{v}) = (S_1^*(\mathbf{v}), \ldots, S_n^*(\mathbf{v}))$ *be a welfare-maximizing allocation. Then, for every bid profile* $\mathbf{b}$ *the following holds:*

$$\sum_{i=1}^{n} \sum_{j \in S_i(\mathbf{b})} p_j(\mathbf{b}) \geq \sum_{i=1}^{n} \sum_{j \in S_i^*(\mathbf{v}) \setminus S_i(\mathbf{b})} b_{ij}$$

Proof. Let $S_{-i}^*(\mathbf{v}) = \bigcup_{j \neq i} S_j^*(\mathbf{v})$. Since payments are non-negative, $S_i(\mathbf{b}) \cap S_{-i}^*(\mathbf{v}) \subseteq S_i(\mathbf{b})$, and each item is sold in a separate second price auction, we get:

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{j \in S_i(\mathbf{b})} p_j(\mathbf{b}) &\geq \sum_{i=1}^{n} \sum_{j \in S_i(\mathbf{b}) \cap S_{-i}^*(\mathbf{v})} p_j(\mathbf{b}) \;=\; \sum_{i=1}^{n} \sum_{j \in S_i(\mathbf{b}) \cap S_{-i}^*(\mathbf{v})} \max_{k \neq i} b_{kj} \\
&= \sum_{i=1}^{n} \sum_{l=1, l \neq i}^{n} \sum_{j \in S_i(\mathbf{b}) \cap S_l^*(\mathbf{v})} \max_{k \neq i} b_{kj} \;\geq\; \sum_{i=1}^{n} \sum_{l=1, l \neq i}^{n} \sum_{j \in S_i(\mathbf{b}) \cap S_l^*(\mathbf{v})} b_{lj} \qquad (1) \\
&= \sum_{l=1}^{n} \sum_{j \in S_l^*(\mathbf{v}) \setminus S_l(\mathbf{b})} b_{lj}
\end{aligned}$$

Inequality (1) holds since b_{lj} is at most the second highest bid on item $j \in S_i(\mathbf{b})$. Notice that the term in (1) considers for each player i all the items she wins in bid profile $\mathbf{b}$, which are allocated to some other player $l \neq i$ in the optimal allocation. Instead, we can change the order of summation and consider for each player l all the items which are allocated to her in the optimal allocation, but not in bid profile $\mathbf{b}$. This accounts for the last equality. □

Incomplete information setting: solution concepts and Bayesian PoA. In an incomplete information setting, player valuations are drawn from a commonly known, possibly correlated, joint distribution $\mathcal{F} \in \Delta(\mathcal{V}_1 \times \ldots \times \mathcal{V}_n)$, and the valuation v_i of each player is a private information which is known only to player i. The strategy of player i is a function $\sigma_i : \mathcal{V}_i \to \mathcal{B}_i$. Let Σ_i denote the strategy space of player i and $\Sigma = \Sigma_1 \times \ldots \times \Sigma_n$ the strategy space of all players. We denote by $\sigma(\mathbf{v}) = (\sigma_1(v_1), \ldots, (\sigma_n(v_n))$ the bid vector given a valuation profile $\mathbf{v}$.

In some cases, we assume that the joint distribution of the valuations is a product distribution, i.e., $\mathcal{F} = \mathcal{F}_1 \times \ldots \times \mathcal{F}_n \in \Delta(\mathcal{V}_1) \times \ldots \times \Delta(\mathcal{V}_n)$. In these cases, each valuation v_i is independently drawn from the commonly known distribution $\mathcal{F}_i \in \Delta(\mathcal{V}_i)$.

The standard equilibrium concepts in the incomplete information setting are the Bayes Nash equilibrium (BNE) and the mixed Bayes Nash equilibrium (MBNE):

Definition 2.5 *(Bayes Nash Equilibrium (BNE)).* A strategy profile σ is a BNE if for any $i \in [n]$, any $v_i \in \mathcal{V}_i$ and any $b_i' \in \mathcal{B}_i$,

$$\mathbb{E}_{\mathbf{v}_{-i}|v_i} \left[u_i(\sigma_i(v_i), \sigma_{-i}(\mathbf{v}_{-i}), v_i)\right] \geq \mathbb{E}_{\mathbf{v}_{-i}|v_i} \left[u_i(b_i', \sigma_{-i}(\mathbf{v}_{-i}), v_i)\right] \qquad (2)$$

Definition 2.6 *(Mixed Bayes Nash Equilibrium (MBNE)).* A randomized strategy profile σ is a MBNE if for any $i \in [n]$, any $v_i \in \mathcal{V}_i$ and any $b_i' \in \mathcal{B}_i$,

$$\mathbb{E}_{\mathbf{v}_{-i}|v_i} \mathbb{E}_{\sigma} \left[u_i(\sigma_i(v_i), \sigma_{-i}(\mathbf{v}_{-i}), v_i)\right] \geq \mathbb{E}_{\mathbf{v}_{-i}|v_i} \mathbb{E}_{\sigma_{-i}} \left[u_i(b_i', \sigma_{-i}(\mathbf{v}_{-i}), v_i)\right]$$

Note that if player valuations are independent, we can omit the conditioning on v_i in Definitions 2.5 and 2.6.

The Bayes Nash price of anarchy is:

$$BPoA = \inf_{\mathcal{F},\, \sigma \in BNE} \frac{\mathbb{E}_{\mathbf{v}}[SW(\sigma(\mathbf{v}), \mathbf{v})]}{\mathbb{E}_{\mathbf{v}}[OPT(\mathbf{v})]}$$

The mixed Bayes Nash price of anarchy is defined similarly w.r.t. MBNE.

2.2. Valuation classes

In what follows we present the valuation functions considered in this paper. As standard, for a valuation v, item j and set S, we denote the marginal value of item j, given set S, as $v(j \mid S)$; i.e., $v(j \mid S) = v(S \cup \{j\}) - v(S)$. In a similar manner, the marginal value of a set S', given a set S, is $v(S' \mid S) = v(S \cup S') - v(S)$. Following are the valuation classes we consider:

unit-demand (UD): A valuation function v is UD if there exist values $v_1, \ldots, v_m$ such that for every set $S \subseteq [m]$, $v(S) = max_{j \in S} v_j$.

submodular (SM): A valuation function v is SM if for every two sets $S \subseteq T \subseteq [m]$ and element $j \notin T$, $v(j \mid S) \geq v(j \mid T)$.
XOS (also known as fractionally subadditive): A valuation function v is XOS if there exists a set $\mathcal{L}$ of additive valuations $\{a_\ell(\cdot)\}_{\ell \in \mathcal{L}}$, such that for every set $S \subseteq [m]$, $v(S) = max_{\ell \in \mathcal{L}} a_\ell(S)$.
subadditive (SA): A valuation function v is SA if for any subsets $S, T \subseteq [m]$, $v(S) + v(T) \geq v(S \cup T)$.
monotone (MON): A valuation function v is MON if $\forall S \subseteq T \subseteq [m]$, $v(S) \leq v(T)$.

A strict containment hierarchy of the above valuation classes is known: $UD \subset SM \subset XOS \subset SA \subset MON$.

2.2.1. α-SM valuation class

Classes of set functions are usually characterized by some convenient properties that make them useful in optimization, characterization, approximation, etc. In practice, the input might only *approximately* adhere to some structural property. To fall within a particular structural property, fairly stringent constraints should be satisfied. The question, then, is whether the guarantees associated with these stringent constraints continue to hold approximately given that these constraints hold approximately. This motivates us to introduce a new class of valuation functions, parameterized by 'how far' they are from submodular valuations:

Definition 2.7 *($\alpha-$**submodular** ($\alpha-$**SM**)).* A valuation function v is α-SM, for $0 < \alpha \leq 1$, if for every two sets $S \subseteq T \subseteq [m]$ and element $j \notin T$, $v(j \mid S) \geq \alpha \cdot v(j \mid T)$.

By definition, a SM valuation is precisely 1-SM. Note that the definitions of α-SM and XOS are incomparable. In particular, there is no $\alpha > 0$, such that every XOS function is α-SM (see Appendix B.1), and for every $0 < \alpha < 1$, there exists an α-SM function that is not XOS (see Appendix B.2).

Lemma 2.2. *For any α-SM function v and any sets S, S': $\sum_{j \in S'} v(j \mid S) \geq \alpha \cdot v(S' \mid S)$*

Proof. Let $S' = \left\{j_1, j_2, \ldots, j_{|S'|}\right\}$. As v is α-SM, we have $v(j_i \mid S) \geq \alpha \;\; v(j_i \mid S \cup \{j_1, \ldots, j_{i-1}\})$ for every $i = 1, \ldots, |S'|$. Therefore, $\sum_{j \in S'} v(j \mid S) = \sum_{i=1}^{|S'|} v(j_i \mid S) \geq \alpha \sum_{i=1}^{|S'|} v(j_i \mid S \cup \{j_1, \ldots, j_{i-1}\}) = \alpha \cdot v(S' \mid S)$. The inequality follows from α-submodularity, and the last equality is due to telescoping sum. □

Lemma 2.3. *If a valuation function, v, is α-SM, then there exists a set $\mathcal{L}$ of additive valuations $\{a^\ell(\cdot)\}_{\ell \in \mathcal{L}}$, such that for every set $S \subseteq [m]$, $v(S) \geq \alpha \cdot max_{\ell \in \mathcal{L}} \left[a^\ell(S)\right]$ and there exists at least one ℓ such that $v(S) = a^\ell(S)$.*

Proof. The proof is an extension of the proof in Lehmann et al. (2006) that any submodular function is XOS. Define $m!$ additive valuations a^ℓ, one for each permutation of the items in $[m]$. Let $a_j^\ell = v(j \mid S_j^\ell)$, where S_j^ℓ is the set of items in permutation ℓ preceding item j. For any permutation ℓ and set $S = \{1, 2, \ldots, k\} \subseteq [m]$ with item j denoting the jth item of S in the permutation ℓ,

$$\begin{aligned} a^\ell(S) = \sum_{j \in S} a_j^\ell \;\; &= \;\; \sum_{j \in S} v(j \mid S_j^\ell) \\ &\leq \sum_{j \in S} \frac{1}{\alpha} \left[v(\{1, 2, \ldots, j\}) - v(\{1, 2, \ldots, j-1\})\right] \;\; = \;\; \frac{1}{\alpha} v(S), \end{aligned}$$

where the inequality follows from the definition of α-SM. For any permutation ℓ in which the items of S are placed first, we have $v(S) = a^\ell(S)$. □

2.3. Smooth auctions

We use a smoothness definition based on Roughgarden (2009) and Roughgarden et al. (2017):

Definition 2.8 *(Smooth auction (based on Roughgarden (2009), Roughgarden et al. (2017))).* An auction is (λ, μ)-smooth for parameters $\lambda, \mu \geq 0$ with respect to a bid space $\mathcal{B}' \subseteq \mathcal{B}_1 \times \ldots \times \mathcal{B}_n$, if for any valuation profile $\mathbf{v} \in \mathcal{V}_1 \times \ldots \times \mathcal{V}_n$ and any bid profile $\mathbf{b} \in \mathcal{B}'$ there exists a bid $b_i^*(\mathbf{v}) \in \mathcal{B}_i$ for each player i, s.t.:

$$\sum_{i \in [n]} u_i(b_i^*(\mathbf{v}), \mathbf{b}_{-i}, v_i) \geq \lambda \cdot OPT(\mathbf{v}) - \mu \cdot SW(\mathbf{b}, \mathbf{v}) \tag{3}$$

It is shown in Roughgarden (2009); Roughgarden et al. (2017) that for every (λ, μ)-smooth auction, the social welfare of any PNE is at least $\frac{\lambda}{1+\mu}$ of the optimal SW. Via extension theorems, this bound extends to CCE in full-information settings and to Bayes NE in settings with incomplete information. These theorems are stated below, and their proofs appear in Appendix C for completeness.

Theorem 2.1. ***(based on Roughgarden (2009), Roughgarden et al. (2017))*** *If an auction is (λ, μ)-smooth with respect to a bid space $\mathcal{B}' \subseteq \mathcal{B}_1 \times \ldots \times \mathcal{B}_n$, then the expected social welfare of any coarse correlated equilibrium, $\mathbf{b} \in \Delta(\mathcal{B}')$, of the auction is at least $\frac{\lambda}{1+\mu}$ of the optimal social welfare.*

Theorem 2.2. ***(based on Roughgarden (2012), Syrgkanis and Tardos (2013))*** *If an auction is (λ, μ)-smooth with respect to a bid space $\mathcal{B}' \subseteq \mathcal{B}_1 \times \ldots \times \mathcal{B}_n$, then for every product distribution $\mathcal{F}$, every mixed Bayes Nash equilibrium, $\sigma : \mathcal{V}_1 \times \ldots \times \mathcal{V}_n \to \Delta(\mathcal{B}')$ has expected social welfare at least $\frac{\lambda}{1+\mu}$ of the expected optimal social welfare.*

A standard assumption in essentially all previous work on the PoA of simultaneous second price item auction (e.g., Christodoulou et al. (2016a), Feldman et al. (2013), Roughgarden (2012), Roughgarden (2009), Bhawalkar and Roughgarden (2011)) is *no overbidding*, meaning that players do not overbid on items they win. Formally,

Definition 2.9 *(No overbidding (NOB)).* Given a valuation profile $\mathbf{v} \in \mathcal{V}_1 \times \ldots \times \mathcal{V}_n$, a bid profile $\mathbf{b} \in \mathcal{B}$ is said to satisfy NOB if for every player i the following holds,

$$\sum_{j \in S_i(\mathbf{b})} b_{ij} \leq v_i(S_i(\mathbf{b}))$$

Theorem 2.3. ***(based on Christodoulou et al. (2016a) and Roughgarden (2009))****: S2PA with XOS valuations is $(1, 1)$-smooth, with respect to bid profiles satisfying NOB.*

Theorem 2.3 implies a lower bound of $\frac{1}{2}$ on the Bayesian PoA of S2PA with XOS valuations. This result is tight, even with respect to unit-demand valuations in full information settings Christodoulou et al. (2016a). We now extend the last result to α-SM valuations.

Theorem 2.4. *S2PA with α-SM valuations is $(\alpha, 1)$-smooth, with respect to bid profiles satisfying NOB.*

Proof. Let $\mathbf{v} \in \mathcal{V}_1 \times \ldots \times \mathcal{V}_n$ be an α-SM valuation profile and let $\mathbf{b}$ be a bid profile satisfying NOB. From Lemma 2.3, for every valuation v_i there exists a set $\{a_i^\ell(\cdot)\}_{\ell \in \mathcal{L}_i}$, such that for every set $S \subseteq [m]$, $v_i(S) \geq \alpha \cdot max_{\ell \in \mathcal{L}_i}\left[a_i^\ell(S)\right]$ and there exists ℓ such that $v_i(S) = a_i^\ell(S)$. Let $S^*(\mathbf{v}) = (S_1^*(\mathbf{v}), \ldots, S_n^*(\mathbf{v}))$ be a welfare maximizing allocation, and let a_i^* be an additive valuation such that $v_i(S_i^*(\mathbf{v})) = a_i^*(S_i^*(\mathbf{v}))$. Consider the following hypothetical deviation for player i: $b_{ij}^* = a_{ij}^*$ if $j \in S_i^*(\mathbf{v})$, and $b_{ij}^* = 0$ otherwise.

Now let us consider the utility of player i when deviating. As $b_{ij}^* = 0$ for every item $j \notin S_i^*(\mathbf{v})$, each such item contributes non-negative utility to i and we can ignore this contribution while lower bounding i's utility under b_i^*. Consider item $j \in S_i^*(\mathbf{v})$. If $a_{ij}^* \geq \max_{k \neq i} b_{kj}$, player i wins item j. Otherwise, i does not win item j, and the term $a_{ij}^* - \max_{k \neq i} b_{kj}$ is non-positive. Since $v_i(S) \geq \alpha \cdot a_i^*(S)$ for every set S, and since $\alpha \leq 1$, we get:

$$\begin{aligned}
&\sum_{i \in [n]} u_i(b_i^*(\mathbf{v}), \mathbf{b}_{-i}, v_i) \\
&\geq \sum_{i \in [n]} \left[\sum_{j \in S_i^*(\mathbf{v}) \cap S_i(b_i^*(\mathbf{v}), \mathbf{b}_{-i})} \left(\alpha \cdot a_{ij}^* - \max_{k \neq i} b_{kj} \right) + \sum_{j \in S_i^*(\mathbf{v}) \setminus S_i(b_i^*(\mathbf{v}), \mathbf{b}_{-i})} \left(a_{ij}^* - \max_{k \neq i} b_{kj} \right) \right] \\
&\geq \sum_{i \in [n]} \sum_{j \in S_i^*(\mathbf{v})} \left(\alpha \cdot a_{ij}^* - \max_{k \neq i} b_{kj} \right) \\
&\geq \alpha \sum_{i \in [n]} v_i(S_i^*(\mathbf{v})) - \sum_{i \in [n]} \sum_{j \in S_i^*(\mathbf{v})} \max_k b_{kj} \\
&\geq \alpha \cdot OPT(\mathbf{v}) - \sum_{i \in [n]} \sum_{j \in S_i(\mathbf{b})} \max_k b_{kj} \\
&= \alpha \cdot OPT(\mathbf{v}) - \sum_{i \in [n]} \sum_{j \in S_i(\mathbf{b})} b_{ij} \\
&\geq \alpha \cdot OPT(\mathbf{v}) - \sum_{i \in [n]} v_i(S_i(\mathbf{b}))
\end{aligned}$$

$$= \alpha \cdot OPT(\mathbf{v}) - SW(\mathbf{b}, \mathbf{v}).$$

The third inequality follows from the choice of a_i^* and by the payment structure of 2nd price. The fourth inequality follows by the fact that all items are allocated in S2PA. Finally, the last inequality follows from NOB. $\square$

3. Revenue guaranteed auctions

Following is the definition of revenue guaranteed auctions. We then discuss the implications of this property in both full information and incomplete information settings.

Definition 3.1 *(Revenue guaranteed auction).* An auction is (γ, δ)-revenue guaranteed for some $0 \le \gamma \le \delta \le 1$ with respect to a bid space $\mathcal{B}' \subseteq \mathcal{B}_1 \times \ldots \times \mathcal{B}_n$, if for any valuation profile $\mathbf{v} \in \mathcal{V}_1 \times \ldots \times \mathcal{V}_n$ and for any bid profile $\mathbf{b} \in \mathcal{B}'$ the revenue of the auction is at least $\gamma \cdot OPT(\mathbf{v}) - \delta \cdot SW(\mathbf{b}, \mathbf{v})$.

3.1. Full information: revenue guaranteed auctions

The following theorem establishes welfare guarantees on every pure bid profile of a (γ, δ)-revenue guaranteed auction in which the sum of player utilities is non-negative.

Theorem 3.1. *If an auction is (γ, δ)-revenue guaranteed with respect to a bid space $\mathcal{B}' \subseteq \mathcal{B}_1 \times \cdots \times \mathcal{B}_n$, then for any pure bid profile $\mathbf{b} \in \mathcal{B}'$, in which the sum of player utilities is non-negative, the social welfare is at least $\frac{\gamma}{1+\delta}$ of the optimal social welfare.*

Proof. Using quasi-linear utilities and non-negative sum of player utilities, we get:

$$0 \le \sum_{i \in [n]} u_i(\mathbf{b}, v_i) = \sum_{i \in [n]} v_i(S_i(\mathbf{b})) - \sum_{i \in [n]} P_i(\mathbf{b}) = SW(\mathbf{b}, \mathbf{v}) - \sum_{i \in [n]} P_i(\mathbf{b})$$

By the (γ, δ)-revenue guaranteed property,

$$\sum_{i \in [n]} P_i(b) \ge \gamma OPT(\mathbf{v}) - \delta SW(b, \mathbf{v}).$$

Putting it all together, we get

$$0 \;\le\; SW(\mathbf{b}, \mathbf{v}) - \sum_{i \in [n]} P_i(\mathbf{b}) \;\le\; (1+\delta) SW(\mathbf{b}, \mathbf{v}) - \gamma OPT(\mathbf{v}) \tag{4}$$

Rearranging, we get: $SW(\mathbf{b}, \mathbf{v}) \ge \frac{\gamma}{1+\delta} OPT(\mathbf{v})$, as required. $\square$

Definition 3.1 considers pure bid profiles, but Theorem 3.1 applies to the more general setting of randomized bid profiles, possibly correlated, as cast in the following extension theorem.

Theorem 3.2. *If an auction is (γ, δ)-revenue guaranteed with respect to a bid space $\mathcal{B}' \subseteq \mathcal{B}_1 \times \ldots \times \mathcal{B}_n$, then for any bid profile $\mathbf{b} \in \Delta(\mathcal{B}')$, in which the sum of the expected utilities of the players is non-negative, the expected social welfare is at least $\frac{\gamma}{1+\delta}$ of the optimal social welfare.*

The proof is identical to the proof of Theorem 3.1, except adding expectation over $\mathbf{b}$ to every term, using the fact the auction is (γ, δ)-revenue guaranteed for every b in the support of $\mathbf{b}$, and using linearity of expectation.

Clearly, in every equilibrium (including CCE) the expected utility of every player is non-negative. It therefore follows that the expected welfare in any CCE is at least $\frac{\gamma}{1+\delta}$ of the optimal social welfare.

For an auction that is both smooth and revenue guaranteed, we give a better bound on the price of anarchy:

Theorem 3.3. *If an auction is (λ, μ)-smooth with respect to a bid space $\mathcal{B}'$ and (γ, δ)-revenue guaranteed with respect to a bid space $\mathcal{B}''$, then the expected social welfare at any CCE $\in \Delta(\mathcal{B}' \cap \mathcal{B}'')$ of the auction is at least $\frac{\lambda+\gamma}{1+\mu+\delta}$ of the optimal social welfare.*

Proof. The proof follows by the proofs of Theorem 2.1 and Theorem 3.2. Let $\mathbf{b} \in \Delta(\mathcal{B}' \cap \mathcal{B}'')$ be a CCE of the auction. The proof of Theorem 2.1 shows that:

$$\sum_{i \in [n]} \mathbb{E}_{\mathbf{b}}[u_i(\mathbf{b}, v_i)] \ge \lambda \cdot OPT(\mathbf{v}) - \mu \cdot \mathbb{E}_{\mathbf{b}}[SW(\mathbf{b}, \mathbf{v})]$$

From Equation (4) we get,

$$\mathbb{E}_{\mathbf{b}}[SW(\mathbf{b},\mathbf{v})] - \sum_{i\in[n]} \mathbb{E}_{\mathbf{b}}[P_i(\mathbf{b})] \le (1+\delta)\cdot \mathbb{E}_{\mathbf{b}}[SW(\mathbf{b},\mathbf{v})] - \gamma \cdot OPT(\mathbf{v})$$

As utilities are quasi-linear, the left hand side of the above two inequalities are equal. Rearranging, we get: $\mathbb{E}_{\mathbf{b}}[SW(\mathbf{b},\mathbf{v})] \ge \frac{\lambda+\gamma}{1+\mu+\delta} OPT(\mathbf{v})$, as required. □

3.2. Incomplete information: extension theorem for revenue guaranteed auctions

In a similar manner to the smoothness extension theorem, we can prove an extension theorem for the revenue guaranteed property, which gives expected welfare guarantees for settings with incomplete information. However, this extension theorem is stronger, in the sense that it holds with respect to correlated distributions and not only product prior distributions.

Theorem 3.4. *If an auction is (γ,δ)-revenue guaranteed with respect to a bid space $\mathcal{B}' \subseteq \mathcal{B}_1 \times \cdots \times \mathcal{B}_n$, then for every joint distribution $\mathcal{F} \in \Delta(\mathcal{V}_1 \times \ldots \times \mathcal{V}_n)$, possibly correlated, and every strategy profile $\sigma : \mathcal{V}_1 \times \ldots \times \mathcal{V}_n \to \Delta(\mathcal{B}')$, in which the expected sum of player utilities is non-negative, the expected social welfare is at least $\frac{\gamma}{1+\delta}$ of the expected optimal social welfare.*

Proof. We give a proof for pure strategies. The proof for mixed strategies follows by adding in a straightforward way another expectation over the random actions chosen in the strategy profile σ. As the utility of each player is quasi-linear and the expected sum of player utilities is non-negative, we use linearity of expectation and get,

$$\begin{aligned} 0 &\le \sum_{i\in[n]} \mathbb{E}_{\mathbf{v}}[u_i(\sigma(\mathbf{v}), v_i)] \\ &= \sum_{i\in[n]} \mathbb{E}_{\mathbf{v}}[v_i(S_i(\sigma(\mathbf{v})))] - \sum_{i\in[n]} \mathbb{E}_{\mathbf{v}}[P_i(\sigma(\mathbf{v}))] \\ &= \mathbb{E}_{\mathbf{v}}[SW(\sigma(\mathbf{v}),\mathbf{v})] - \sum_{i\in[n]} \mathbb{E}_{\mathbf{v}}[P_i(\sigma(\mathbf{v}))] \end{aligned}$$

By the (γ,δ)-revenue guaranteed property, for each v in the support of $\mathbf{v}$,

$$\sum_{i\in[n]} P_i(\sigma(v)) \ge \gamma \cdot OPT(v) - \delta \cdot SW(\sigma(v), v)$$

Putting it all together, we get,

$$\begin{aligned} 0 &\le \mathbb{E}_{\mathbf{v}}[SW(\sigma(\mathbf{v}),\mathbf{v})] - \sum_{i\in[n]} \mathbb{E}_{\mathbf{v}}[P_i(\sigma(\mathbf{v}))] \\ &\le \mathbb{E}_{\mathbf{v}}[SW(\sigma(\mathbf{v}),\mathbf{v})] - \mathbb{E}_{\mathbf{v}}[\gamma OPT(\mathbf{v}) - \delta SW(\sigma(\mathbf{v}),\mathbf{v})] && (5) \\ &= (1+\delta)\cdot \mathbb{E}_{\mathbf{v}}[SW(\sigma(\mathbf{v}),\mathbf{v})] - \gamma \cdot \mathbb{E}_{\mathbf{v}}[OPT(\mathbf{v})] && (6) \end{aligned}$$

Rearranging, we get: $\mathbb{E}_{\mathbf{v}}[SW(\sigma(\mathbf{v}),\mathbf{v})] \ge \frac{\gamma}{1+\delta}\mathbb{E}_{\mathbf{v}}[OPT(\mathbf{v})]$, as required. □

As the expected utility of each player is non-negative at any equilibrium strategy profile, we infer that if an auction is (γ,δ)-revenue guaranteed with respect to a bid space $\mathcal{B}'$, then for every joint distribution $\mathcal{F} \in \Delta(\mathcal{V}_1 \times \ldots \times \mathcal{V}_n)$, possibly correlated, the expected social welfare at any mixed Bayes Nash equilibrium, $\sigma : \mathcal{V}_1 \times \ldots \times \mathcal{V}_n \to \Delta(\mathcal{B}')$, is at least $\frac{\gamma}{1+\delta}$ of the expected optimal social welfare.

For an auction that is both smooth and revenue guaranteed, we give a better bound on the price of anarchy, if the joint distribution $\mathcal{F}$ is a product distribution:

Theorem 3.5. *If an auction is (λ,μ)-smooth with respect to a bid space $\mathcal{B}'$ and (γ,δ)-revenue guaranteed with respect to a bid space $\mathcal{B}''$, then for every product distribution $\mathcal{F}$, every mixed Bayes Nash equilibrium, $\sigma : \mathcal{V}_1 \times \ldots \times \mathcal{V}_n \to \Delta(\mathcal{B}' \cap \mathcal{B}'')$, has expected social welfare at least $\frac{\lambda+\gamma}{1+\mu+\delta}$ of the expected optimal social welfare.*

Proof. We give a proof for pure Bayes Nash equilibrium. The proof for mixed Bayes Nash equilibrium follows by adding in a straightforward way another expectation over the random actions chosen in the strategy profile σ. The proof follows by the proofs of Theorem 2.2 and Theorem 3.4.

The proof of Theorem 2.2 shows that:

$$\mathbb{E}_{\mathbf{v}}\left[\sum_{i\in[n]} u_i(\sigma(\mathbf{v}), v_i)\right] \ge \lambda \cdot \mathbb{E}_{\mathbf{v}}[OPT(\mathbf{v})] - \mu \cdot \mathbb{E}_{\mathbf{v}}[SW(\sigma(\mathbf{v}),\mathbf{v})]$$

From Equation (6) we get,

$$\mathbb{E}_{\mathbf{v}}[SW(\sigma(\mathbf{v}),\mathbf{v})] - \sum_{i\in[n]} \mathbb{E}_{\mathbf{v}}[P_i(\sigma(\mathbf{v}))] \le (1+\delta)\cdot \mathbb{E}_{\mathbf{v}}[SW(\sigma(\mathbf{v}),\mathbf{v})] - \gamma \cdot \mathbb{E}_{\mathbf{v}}[OPT(\mathbf{v})]$$

As utilities are quasi-linear, the left hand side of the above two inequalities are equal. Rearranging, we get: $\mathbb{E}_{\mathbf{v}}[SW(\sigma(\mathbf{v}),\mathbf{v})] \ge \frac{\lambda+\gamma}{1+\mu+\delta}\mathbb{E}_{\mathbf{v}}[OPT(\mathbf{v})]$, as required. □

Remark. Note that we defined revenue guaranteed auctions for full information with respect to a bid space $\mathcal{B}'$, and then used an extension theorem to prove positive results for incomplete information on strategy space $\sigma : \mathcal{V}_1 \times \ldots \times \mathcal{V}_n \to \Delta(\mathcal{B}')$. A different approach is to add an incomplete information definition to revenue guaranteed auctions with respect to a strategy space Σ' and by that get positive results in incomplete information for a wider strategy space. Moreover, there might be auctions that are revenue guaranteed only in expectation and hence do not fall into the current definition. We give such definitions and corresponding theorems in Appendix F.

4. Simultaneous second price auctions with no-underbidding

We begin this section by defining what it means to *underbid* on an item. Consider a bid profile $\mathbf{b}$. Let $\mathbf{b}_{-j}$ be the bids of all bidders on all items except j, and let b_{-ij} be the bids on item j of all players, except player i.

Definition 4.1 *(item underbidding).* Fix $\mathbf{b}_{-j}$. Player i is said to *underbid* on item j if: $b_{ij} < v_i(j \mid S_i(\mathbf{b}_{-j}))$, where $S_i(\mathbf{b}_{-j}) = \{k \mid k \neq j, b_{ik} = \max_l \{b_{lk}\}\}$.

That is, we say that player i underbids on item j in a bid profile $\mathbf{b}$ if i's bid on item j is smaller than the marginal valuation of j with respect to the set of items other than j won by i.

We next show that underbidding is weakly dominated in a precise sense that we define next.

Definition 4.2 *(weakly dominated).* A bid b'_{ij} is *weakly dominated* by bid b_{ij}, with respect to $\mathbf{b}_{-j}$, if the following two conditions hold:

1. $u_i(b_{ij}, b_{-ij}, \mathbf{b}_{-j}; v_i) \ge u_i(b'_{ij}, b_{-ij}, \mathbf{b}_{-j}; v_i)$, for every b_{-ij}
2. There exists b_{-ij} such that the inequality in (1) holds strictly.

The following lemma shows that underbidding on an item in a bid profile is weakly dominated by bidding its marginal value.

Lemma 4.1. *In S2PA, for every player i, every item j, and every bid profile $\mathbf{b}_{-j}$, underbidding on item j is weakly dominated by bidding $b_{ij} = v_i(j \mid S_i(b_{-j}))$, with respect to $\mathbf{b}_{-j}$.*

Proof. Fix $\mathbf{b}_{-j}$, and denote $p_j = \max_{l\neq i}\{b_{lj}\}$. Let b'_{ij} be an underbidding bid on item j by player i, i.e., $b'_{ij} < v_i(j \mid S_i(\mathbf{b}_{-j}))$. We first show that $u_i(b_{ij}, b_{-ij}, \mathbf{b}_{-j}; v_i) \ge u_i(b'_{ij}, b_{-ij}, \mathbf{b}_{-j}; v_i)$ for every b_{-ij}.

- If $b'_{ij} \ge p_j$, player i wins item j under both b'_{ij} and b_{ij}, pays p_j on item j in both cases, thus $u_i(b_{ij}, b_{-ij}, \mathbf{b}_{-j}; v_i) = u_i(b'_{ij}, b_{-ij}, \mathbf{b}_{-j}; v_i)$.
- If $b_{ij} < p_j$, player i doesn't win item j under both b'_{ij} and b_{ij}, pays 0 on item j in both cases, thus $u_i(b_{ij}, b_{-ij}, \mathbf{b}_{-j}; v_i) = u_i(b'_{ij}, b_{-ij}, \mathbf{b}_{-j}; v_i)$.
- If $b_{ij} \ge p_j$ but $b'_{ij} < p_j$, player i wins item j under b_{ij} and pays p_j on item j, but she doesn't win item j under b'_{ij}. As $v_i(j \mid S_i(b_{-j})) = b_{ij} \ge p_j$, we have $u_i(b_{ij}, b_{-ij}, \mathbf{b}_{-j}; v_i) \ge u_i(b'_{ij}, b_{-ij}, \mathbf{b}_{-j}; v_i)$.

Next, we show that there exists b_{-ij} such that $u_i(b_{ij}, b_{-ij}, \mathbf{b}_{-j}; v_i) > u_i(b'_{ij}, b_{-ij}, \mathbf{b}_{-j}; v_i)$. Let $\epsilon > 0$ be such that $b_{ij} = b'_{ij} + \epsilon$. Consider b_{-ij} and player $l \neq i$ such that $b_{lj} = b'_{ij} + \frac{\epsilon}{2} = p_j$. Then, player i doesn't win item j under b'_{ij}, but she does win item j under b_{ij} with a payment of $b'_{ij} + \frac{\epsilon}{2} < b_{ij} = v_i(j|S_i(b_{-j}))$ on item j. Therefore, $u_i(b_{ij}, b_{-ij}, \mathbf{b}_{-j}; v_i) > u_i(b'_{ij}, b_{-ij}, \mathbf{b}_{-j}; v_i)$. □

Motivated by the above analysis, we next define the notion of item no underbidding (iNUB):

Definition 4.3 *(Item No-UnderBidding (iNUB)).* Given a valuation profile $\mathbf{v} \in \mathcal{V}_1 \times \ldots \times \mathcal{V}_n$, we say that a bid profile $\mathbf{b} \in \mathcal{B}$ satisfies iNUB if for every player i and every item $j \in [m]\backslash S_i(\mathbf{b})$ it holds that: $b_{ij} \ge v_i(j \mid S_i(\mathbf{b}))$.

We also define the following notion of set no underbidding (sNUB), as follows:

Definition 4.4 *(Set No underbidding (sNUB)).* Given a valuation profile $\mathbf{v} \in \mathcal{V}_1 \times \ldots \times \mathcal{V}_n$, we say that a bid profile $\mathbf{b} \in \mathcal{B}$ satisfies sNUB if for every player i and every set $T \subseteq [m] \setminus S_i(\mathbf{b})$, it holds that $\sum_{j \in T} b_{ij} \geq v_i(T \mid S_i(\mathbf{b}))$.

One downside of the above notion is that while a PNE satisfying iNUB/sNUB exists for some valuations (e.g., unit-demand), it may not exist generally. To circumvent the existence problem, we introduce slightly weaker versions of iNUB and sNUB below.

Definition 4.5 *(weak Item No-UnderBidding (weak iNUB)).* Given a valuation profile $\mathbf{v} \in \mathcal{V}_1 \times \ldots \times \mathcal{V}_n$, we say that a bid profile $\mathbf{b} \in \mathcal{B}$ satisfies weak iNUB if there exists a welfare maximizing allocation, $S^*(\mathbf{v}) = (S_1^*(\mathbf{v}), \ldots, S_n^*(\mathbf{v}))$, such that for every player i and every item $j \in S_i^*(\mathbf{v}) \setminus S_i(\mathbf{b})$ it holds that: $b_{ij} \geq v_i(j \mid S_i(\mathbf{b}))$.

Definition 4.6 *(weak Set No underbidding (weak sNUB)).* Given a valuation profile $\mathbf{v} \in \mathcal{V}_1 \times \ldots \times \mathcal{V}_n$, we say that a bid profile $\mathbf{b} \in \mathcal{B}$ satisfies weak sNUB if there exists a welfare maximizing allocation, $S^*(\mathbf{v}) = (S_1^*(\mathbf{v}), \ldots, S_n^*(\mathbf{v}))$, such that for every player i, it holds that

$$\sum_{j \in S'} b_{ij} \geq v_i(S' \mid S_i(\mathbf{b})), \quad \text{where} \quad S' = S_i^*(\mathbf{v}) \setminus S_i(\mathbf{b}).$$

As implied by its name, a weak iNUB (respectively, weak sNUB) is weaker than iNUB (resp. sNUB). The main advantage of the weaker notion is that it allows for existence quite generally (see below). A potential disadvantage of the weak notion is that it may lead to a worse price of anarchy (Clearly, every positive result regarding the price of anarchy with respect to weak sNUB carries over to sNUB, by definition, but not vice versa). As it turns out, however, all of our price of anarchy bounds for the stronger notions carry over to the weaker notions; indeed, the instances that realize the worst case price of anarchy among all weak sNUB (resp., weak iNUB) satisfy also the stronger notion. For this reason, we state all of our results for the weaker notions, where existence is guaranteed.

Relation between the no underbidding notions. By definition, any bid profile that satisfies sNUB, also satisfies iNUB. We show below that when valuations are submodular, a bid profile satisfies iNUB if and only if it satisfies sNUB. For unit-demand bidders, this is also true for the weaker notions of iNUB and sNUB; i.e., weak iNUB and weak sNUB coincide (as one can assume w.l.o.g. that every bidder receives a single item in an optimal allocation). However, for more general valuations this is not necessarily true. In Section 5 we show that if valuations are submodular, every bid profile that satisfies weak iNUB, also satisfies weak sNUB. The opposite is not necessarily true, as demonstrated in Appendix D.

The following theorem shows that weak sNUB is a powerful property.

Theorem 4.1. *S2PA with monotone valuation functions is* $(1, 1)$*-revenue guaranteed with respect to bid profiles satisfying weak sNUB.*

Proof. In what follows, the first inequality follows by Lemma 2.1, and the second inequality follows by the fact that $\mathbf{b}$ satisfies weak sNUB. The last inequality follows by monotonicity of valuations.

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{j \in S_i(\mathbf{b})} p_j(\mathbf{b}) &\geq \sum_{i=1}^{n} \sum_{j \in S_i^*(\mathbf{v}) \setminus S_i(\mathbf{b})} b_{ij} \\
&\geq \sum_{i=1}^{n} \left[v_i\left(S_i(\mathbf{b}) \cup \left(S_i^*(\mathbf{v}) \setminus S_i(\mathbf{b}) \right) \right) - v_i\left(S_i(\mathbf{b}) \right) \right] \\
&= \sum_{i=1}^{n} [v_i(S_i(\mathbf{b}) \cup S_i^*(\mathbf{v})) - v_i(S_i(\mathbf{b}))] \\
&\geq \sum_{i=1}^{n} [v_i(S_i^*(\mathbf{v})) - v_i(S_i(\mathbf{b}))] \\
&= OPT(\mathbf{v}) - SW(\mathbf{b}, \mathbf{v}) \quad \square
\end{aligned}$$

The following corollary follows directly by Theorems 4.1 and 3.4.

Corollary 4.1. *In an S2PA with monotone valuations, for every joint distribution $\mathcal{F} \in \Delta(\mathcal{V}_1 \times \ldots \times \mathcal{V}_n)$, possibly correlated, every mixed Bayes Nash equilibrium that satisfies weak sNUB has expected social welfare at least $\frac{1}{2}$ of the expected optimal social welfare.*

Remark. In this section we give a full information definition of weak no-underbidding bid profiles (weak sNUB) and prove Theorem 4.1 accordingly. In Appendix F we give an incomplete information definition of weak no-underbidding strategy profiles, which requires weak no-underbidding in expectation. This broader definition, together with a broader definition of incomplete information revenue guaranteed auctions, allows us to get positive results in incomplete information setting for a wider strategy space.

5. S2PA with submodular valuations

In this section we study S2PA with submodular (and α-submodular) valuations. We first show that for this class of valuations, the notion of weak iNUB suffices for establishing positive results.

Theorem 5.1. *Every S2PA with α-SM valuations is (α, α)-revenue guaranteed with respect to bid profiles satisfying weak iNUB.*

Proof. In what follows, the first inequality follows by Lemma 2.1, and the second inequality follows by the fact that $\mathbf{b}$ satisfies weak iNUB.

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j\in S_i(\mathbf{b})} p_j(\mathbf{b}) &\geq \sum_{i=1}^{n}\sum_{j\in S_i^*(\mathbf{v})\setminus S_i(\mathbf{b})} b_{ij} \\
&\geq \sum_{i=1}^{n}\sum_{j\in S_i^*(\mathbf{v})\setminus S_i(\mathbf{b})} v_i(\, j \mid S_i(\mathbf{b})\,) \\
&= \sum_{i=1}^{n}\sum_{j\in S_i^*(\mathbf{v})} v_i(\, j \mid S_i(\mathbf{b})\,) && (7)\\
&\geq \sum_{i=1}^{n}\alpha\cdot v_i(\, S_i^*(\mathbf{v}) \mid S_i(\mathbf{b})\,) && (8)\\
&\geq \sum_{i=1}^{n}\alpha\cdot [\, v_i(\, S_i^*(\mathbf{v})\,) - v_i(\, S_i(\mathbf{b})\,)\,] && (9)\\
&= \alpha\cdot OPT(\mathbf{v}) - \alpha\cdot SW(\mathbf{b},\mathbf{v})
\end{aligned}$$

Equality (7) is due to the fact that $v_i(j \mid S_i(\mathbf{b})) = 0$ for every $j \in S_i(\mathbf{b})$. Inequality (8) follows from Lemma 2.2, and Inequality (9) is due to monotonicity of valuations. □

An immediate corollary from Theorems 3.4 and 5.1 is:

Corollary 5.1. *In an S2PA with α-SM valuations, for every joint distribution $\mathcal{F} \in \Delta(\mathcal{V}_1 \times \ldots \times \mathcal{V}_n)$, possibly correlated, and every strategy profile σ that satisfies weak iNUB for which the expected sum of player utilities is non-negative, the expected social welfare is at least $\frac{\alpha}{1+\alpha}$ of the expected optimal social welfare. In particular, for SM valuations (where $\alpha = 1$), we get at least $\frac{1}{2}$ of the expected optimal social welfare.*

The $1/2$ bound for submodular valuations is tight, even for iNUB bid profile, even with respect to unit-demand valuations and even in equilibrium, as shown in the following proposition.

Proposition 5.1. *There exists an S2PA with unit-demand valuations that admits a PNE bid profile that satisfies iNUB, where the social welfare in equilibrium is $\frac{1}{2}$ of the optimal social welfare.*

Proof. Consider an S2PA with two unit demand players and 2 items, $\{x, y\}$, where $v_1(x) = 2$, $v_1(y) = 1$, $v_2(x) = 1$ and $v_2(y) = 2$. An optimal allocation gives item x to player 1 and item y to player 2, for a welfare of 4. Consider the following bid profile $\mathbf{b}$: $b_{1x} = 1$, $b_{1y} = 100$, $b_{2x} = 100$ and $b_{2y} = 1$. Player 1 wins item y for a price of 1, and player 2 wins item x for a price of 1. It is easy to see that $\mathbf{b}$ is a PNE that satisfies iNUB. The social welfare of this equilibrium is 2, which is $\frac{1}{2}$ of the optimal social welfare. □

We next show that when valuations are submodular, a bid profile satisfies iNUB if and only if it satisfies sNUB.

Proposition 5.2. *For every SM valuation $\mathbf{v}$, a bid profile $\mathbf{b}$ satisfies iNUB if and only if it satisfies sNUB.*

Proof. If $\mathbf{b}$ satisfies sNUB, then by Definitions 4.4 and 4.3 it also satisfies iNUB. If $\mathbf{b}$ satisfies iNUB, then by Definition 4.3, for every player i and every item $j \in [m] \backslash S_i(\mathbf{b})$, it holds that $b_{ij} \geq v_i(j \mid S_i(\mathbf{b}))$. Therefore, for every player i and every set $T \subseteq [m] \backslash S_i(\mathbf{b})$,

$$\sum_{j \in T} b_{ij} \geq \sum_{j \in T} v_i(j \mid S_i(\mathbf{b})) \geq v_i(T \mid S_i(\mathbf{b})),$$

where the last inequality follows by Lemma 2.2 for $\alpha = 1$. Hence, $\mathbf{b}$ also satisfies sNUB. □

The following proposition shows that for submodular valuations, weak iNUB implies weak sNUB. The opposite direction does not necessarily hold, as demonstrated in Appendix D.

Proposition 5.3. *For every SM valuation* $\mathbf{v}$*, every bid profile* $\mathbf{b}$ *that satisfies weak iNUB also satisfies weak sNUB.*

Proof. By weak iNUB, for every item $j \in S_i^*(\mathbf{v}) \setminus S_i(\mathbf{b})$, it holds that $b_{ij} \geq v_i(j \mid S_i(\mathbf{b}))$. It follows that

$$\sum_{j \in S_i^*(\mathbf{v}) \setminus S_i(\mathbf{b})} b_{ij} \geq \sum_{j \in S_i^*(\mathbf{v}) \setminus S_i(\mathbf{b})} v_i(j \mid S_i(\mathbf{b})) \geq v_i(S_i^*(\mathbf{v}) \setminus S_i(\mathbf{b}) \mid S_i(\mathbf{b})),$$

where the last inequality follows by Lemma 2.2 for $\alpha = 1$. □

Therefore, Theorem 4.1 and Corollary 4.1 also apply to every SM valuation that satisfies weak iNUB. That is,

Corollary 5.2. *S2PA with submodular valuation functions is* $(1, 1)$*-revenue guaranteed with respect to bid profiles satisfying weak iNUB.*

Corollary 5.3. *In an S2PA with submodular valuations, for every joint distribution* $\mathcal{F} \in \Delta(\mathcal{V}_1 \times \ldots \times \mathcal{V}_n)$*, possibly correlated, every mixed Bayes Nash equilibrium that satisfies weak iNUB has expected social welfare at least* $\frac{1}{2}$ *of the expected optimal social welfare.*

For bid profiles that satisfy both weak iNUB and NOB, we get a better bound as a direct corollary from Theorems 5.1, 2.4, and 3.5.

Corollary 5.4. *In an S2PA with* α*-SM valuations, for every product distribution* $\mathcal{F}$*, every mixed Bayes Nash equilibrium that satisfies both NOB and weak iNUB has expected social welfare at least* $\frac{2\alpha}{2+\alpha}$ *of the expected optimal social welfare. In particular, for SM valuations (where* $\alpha = 1$*) this amounts to at least* $\frac{2}{3}$ *of the expected optimal social welfare, and this is tight.*

The bound of 2/3 for submodular valuations is tight even for iNUB bid profile, and even with respect to a PNE with unit-demand valuations. This is shown in Example 1.2 in Section 1.

5.1. Existence of PNE

The following proposition shows that when valuations are unit-demand, there always exists a PNE bid profile which satisfies both NOB and sNUB.

Proposition 5.4. *In S2PA with unit-demand valuations there always exists a PNE that satisfies both NOB and sNUB.*

Proposition 5.5 shows that a PNE bid profile which satisfies both NOB and sNUB may not always exist for submodular valuations.

Proposition 5.5. *There exists an instance with submodular valuations that has no S2PA PNE bid profile that satisfies both NOB and sNUB.*

We next show that for submodular valuations there always exists a PNE bid profile which satisfies sNUB.

Proposition 5.6. *In S2PA with submodular valuations there always exists a PNE that satisfies sNUB.*

Proof. Let $S^*(\mathbf{v}) = (S_1^*(\mathbf{v}), \ldots, S_n^*(\mathbf{v}))$ be a welfare maximizing allocation. Consider the bid profile $\mathbf{b}$ in which $b_{ij} = \infty$ if $j \in S_i^*$ and $b_{ij} = v_i(j \mid S_i^*)$ otherwise. We show that $S^*(\mathbf{v})$ is a legal outcome of $\mathbf{b}$, and that $\mathbf{b}$ is a PNE which satisfies sNUB.

Notice that for every i and every $j \in S_i^*$, $b_{ij} = \infty \geq max_{l \neq i} v_l(j \mid S_l^*) = max_{l \neq i} b_{lj}$. Therefore, $S^*(\mathbf{v})$ is a legal outcome of $\mathbf{b}$. Moreover, as valuations are submodular, one can easily verify that $\mathbf{b}$ satisfies sNUB.

It remains to show that $\mathbf{b}$ is a PNE, i.e., no player can gain from a unilateral deviation. As $b_{ij} = \infty$ for every i and every $j \in S_i^*$, no player i can gain from adding an item $j \notin S_i(\mathbf{b})$. Hence, we only need to show that no player can gain from discarding one or more items, i.e., for every i and every $T \subseteq S_i^*$, $u_i(S_i^*) \geq u_i(S_i^* \backslash T)$. Alternatively, we need to show that for every i and every $T \subseteq S_i^*$, $v_i(T \mid S_i^* \backslash T) \geq \sum_{j \in T} max_{l \neq i} b_{lj}$. Let $T = \left\{j_1, j_2, \ldots, j_{|T|}\right\} \subseteq S_i^*$ and let $l(j) = argmax_{l \neq i} b_{lj}$ for every $j \in T$. Then, from submodularity and optimality of S^*, $v_i(T \mid S_i^* \backslash T) = \sum_{k=1}^{|T|} v_i(j_k \mid S_i^* \backslash \left\{j_k, j_{k+1}, \ldots, j_{|T|}\right\}) \geq \sum_{k=1}^{|T|} v_i(j_k \mid S_i^* \backslash j_k) \geq \sum_{k=1}^{|T|} v_{l(j_k)}(j_k \mid S_{l(j_k)}^*) = \sum_{j \in T} max_{l \neq i} b_{lj}$, which completes the proof. □

In the next section, we show that every S2PA with XOS valuations admits a PNE that satisfies both NOB and weak sNUB (Theorem 6.1). Since every submodular valuation is XOS, the existence result applies also to submodular valuations. Actually, the proof of Theorem 6.1 shows that every S2PA with submodular valuations admits a PNE that satisfies both NOB and weak iNUB.

6. S2PA with XOS valuations

6.1. XOS valuations under iNUB

For XOS valuations, iNUB does not imply sNUB nor weak sNUB, thus iNUB does not lead automatically to PoA lower bounds. Indeed, Example 6.1 shows an instance of S2PA with XOS valuations that admits a PNE bid profile that satisfies iNUB, with social welfare that is only a $\frac{2}{m}$ fraction of the optimal social welfare.

Example 6.1. Consider an S2PA with two players, with the following XOS valuation functions v_1 and v_2, respectively, over m items:

$$v_1(S) = max\left\{a^1(S), a^2(S)\right\},$$
$$v_2(S) = max\left\{a^3(S), a^4(S)\right\}$$

where:

$$a^1 = (a_1^1, a_2^1, \ldots, a_m^1) = (2, 2, 0, 0, 0, 0, \ldots, 0),$$
$$a^2 = (a_1^2, a_2^2, \ldots, a_m^2) = (0, 0, 1, 1, 0, 0, \ldots, 0),$$
$$a^3 = (a_1^3, a_2^3, \ldots, a_m^3) = (0, 0, 2, 2, 2, 2, \ldots, 2),$$
$$a^4 = (a_1^4, a_2^4, \ldots, a_m^4) = (1, 1, 0, 0, 0, 0, \ldots, 0).$$

The optimal allocation has welfare $2m$, giving the first two items to player 1 and the last $m-2$ items to player 2. Consider the following bid profile $\mathbf{b} = (b_1, b_2)$, where:

$$b_1 = (0, 0, 2, 2, 2, 2, \ldots, 2)$$
$$b_2 = (2, 2, 0, 0, 0, 0, \ldots, 0)$$

Player 2 wins the first two items and player 1 wins the last $m-2$ items. It is easy to see that $\mathbf{b}$ is an equilibrium which satisfies iNUB. The social welfare of this equilibrium is 4, which is $\frac{2}{m}$ of the optimal social welfare.

6.2. XOS valuations under weak sNUB

As Theorem 4.1 and Corollary 4.1 apply for arbitrary monotone valuation functions, the PoA is at least $\frac{1}{2}$ with respect to bid profiles satisfying weak sNUB. An immediate corollary from Theorems 2.3, 4.1 and 3.5 is:

Corollary 6.1. *In an S2PA with XOS valuations, for every product distribution $\mathcal{F}$, every mixed Bayes Nash equilibrium that satisfies both NOB and weak sNUB has expected social welfare at least $\frac{2}{3}$ of the expected optimal social welfare.*

As in the case of submodular valuations, this result is tight even for sNUB bid profile (see Example 1.2).

6.3. Existence of PNE

The following proposition shows that a PNE bid profile which satisfies sNUB may not always exist for XOS valuations.

Proposition 6.1. *There exists an instance with XOS valuations that has no S2PA PNE bid profile which satisfies sNUB.*

Proof. Consider a setting with five identical items and two players. Player 1's valuation is XOS[3] where,

$$v_1(S) = \begin{cases} 0 & |S| = 0 \\ 6 & |S| = 1 \\ 7 & |S| = 2 \\ 8 & 3 \leq |S| \leq 4 \\ 10 & |S| = 5 \end{cases}$$

and player 2 is unit-demand, where $v_2(S) = 1.9$ for every non-empty bundle. We show that no allocation is a PNE of a bid profile that satisfies sNUB. We first show that, under the sNUB assumption, the utility of player 2 is negative for each non-empty bundle allocation:

- $|S_1| = 0$, $|S_2| = 5$. From sNUB, for every $j \in [m]$, $b_{1j} \geq v_1(1) = 6$. Therefore, $u_2(|S_2|) = v_2(5) - 5 \cdot b_{1j} \leq 1.9 - 5 \cdot 6 < 0$.
- $|S_1| = 5 - |S_2|$, $3 \leq |S_2| \leq 4$. From sNUB, for every $j \in S_2$, $b_{1j} \geq v_1(|S_1| + 1) - v_1(|S_1|) = 1$. Therefore, $u_2(|S_2|) = v_2(|S_2|) - |S_2| \cdot b_{1j} \leq 1.9 - 3 \cdot 1 < 0$.
- $|S_1| = 3$, $|S_2| = 2$. From sNUB, $\sum_{j \in S_2} b_{1j} \geq v_1(5) - v_1(3) = 10 - 8 = 2$. Therefore, $u_2(|S_2|) = v_2(2) - \sum_{j \in S_2} b_{1j} \leq 1.9 - 2 < 0$.
- $|S_1| = 4$, $|S_2| = 1$. From sNUB, $b_{1j} \geq v_1(5) - v_1(4) = 10 - 8 = 2$, for $j \in S_2$. Therefore, $u_2(|S_2|) = v_2(1) - b_{1j} \leq 1.9 - 2 < 0$.

Now consider the allocation $|S_1| = 5$ and $|S_0| = 0$. From sNUB, for every $j \in [m]$, $b_{2j} \geq v_2(1) = 1.9$. As bidder 1 gets an average added value of 1 for each of the last four items and pays at least 1.9 for each, she can gain by discarding these four last items. Therefore, the allocation in which bidder 1 gets all the items cannot be a PNE of a bid profile that satisfies sNUB. □

In contrast to the above result, the following theorem shows that under the weaker condition of weak sNUB, there always exists a PNE that satisfies weak sNUB; moreover there exists such a PNE satisfying both weak sNUB and NOB.

Theorem 6.1. *In S2PA with XOS valuations there always exists a PNE that satisfies both NOB and weak sNUB.*

Proof. Christodoulou et al. (2016a) showed that every S2PA with XOS valuations admits a PNE satisfying NOB. We show that the same PNE satisfies weak sNUB as well. Let $S^*(\mathbf{v}) = (S_1^*(\mathbf{v}), \ldots, S_n^*(\mathbf{v}))$ be a welfare maximizing allocation, and let a_i^* be an additive valuation such that $v_i(S_i^*(\mathbf{v})) = \sum_{j \in S_i^*(\mathbf{v})} a_{ij}^*$. Consider the bid profile in which every player bids according to the maximizing additive valuation with respect to her set $S_i^*(\mathbf{v})$, i.e., $b_{ij} = a_{ij}^*$ for every $j \in S_i^*(\mathbf{v})$ and $b_{ij} = 0$ otherwise. One can easily verify that this bid profile is a PNE that satisfies NOB. It thus remains to show that it also satisfies weak sNUB. Recall that weak sNUB imposes restrictions on the bid values of the set $S' = S_i^*(\mathbf{v}) \setminus S_i(\mathbf{b})$. Under the above bid profile we have $S_i(\mathbf{b}) = S_i^*(\mathbf{v})$, i.e., $S' = \emptyset$ and weak sNUB holds trivially. □

7. S2PA with subadditive valuations

Recall that S2PA with arbitrary monotone valuations is $(1, 1)$-revenue guaranteed with respect to bid profiles that satisfy weak sNUB (Theorem 4.1). Hence, the Bayesian PoA for equilibria satisfying weak sNUB is at least $\frac{1}{2}$ and this bound is tight, even for sNUB bid profiles (Proposition 5.1).

For subadditive valuations, Bhawalkar and Roughgarden (2011) showed that the social welfare of any PNE (if it exists) satisfying *strong* NOB is at least $\frac{1}{2}$ of the optimal social welfare.[4] They also showed that this bound is tight. In their proof, they used the fact that the revenue is non-negative. Under our weak sNUB condition, the auction is $(1, 1)$-revenue guaranteed, implying that the revenue is lower bounded by $OPT(\mathbf{v}) - SW(\mathbf{b}, \mathbf{v})$. Plugging this lower bound into their proof, we get:

Theorem 7.1. *In an S2PA with subadditive valuations and at least one PNE that satisfies both strong NOB and weak sNUB, the social welfare of such a PNE is at least $\frac{2}{3}$ of the expected optimal social welfare.*

Proof. Bhawalkar and Roughgarden (2011) showed that for subadditive valuations the following holds for any PNE, $\mathbf{b}$, satisfying strong NOB, if exists[5]

[3] This valuation appears in Figure 1 of Tomer et al. (2020).

[4] In the *strong* no-overbidding assumption the sum of bids on *any* set of items does not exceed the value of that set, i.e., for every i and every subset $S \subseteq [m]$, $\sum_{j \in S} b_{ij} \leq v_i(S)$.

[5] This may look as a smoothness argument. However, while the hypothetical deviation considered in the smoothness proof depends on $\mathbf{v}$ but not on $\mathbf{b}$, the proof in Bhawalkar and Roughgarden (2011) invokes the Nash equilibrium hypothesis for player i with an hypothetical deviation that depends on the bid vectors $\mathbf{b}_{-i}$ of the other players.

$$\sum_{i\in[n]} u_i(\mathbf{b}, v_i) \geq OPT(\mathbf{v}) - SW(\mathbf{b}, \mathbf{v})$$

Since we assume that $\mathbf{b}$ also satisfies weak sNUB, we have that:

$$\sum_{i=1}^{n} \sum_{j\in S_i(\mathbf{b})} p_j(\mathbf{b}) \geq OPT(\mathbf{v}) - SW(\mathbf{b}, \mathbf{v})$$

As utilities are quasi-linear, we get,

$$\begin{aligned}\sum_{i\in[n]} u_i(\mathbf{b}, v_i) &= SW(\mathbf{b}, \mathbf{v}) - \sum_{i=1}^{n} \sum_{j\in S_i(\mathbf{b})} p_j(\mathbf{b}) \\ &\leq SW(\mathbf{b}, \mathbf{v}) - [OPT(\mathbf{v}) - SW(\mathbf{b}, \mathbf{v})] \\ &= 2 \cdot SW(\mathbf{b}, \mathbf{v}) - OPT(\mathbf{v})\end{aligned}$$

Putting it all together, we get:

$$\begin{aligned}2 \cdot SW(\mathbf{b}, \mathbf{v}) - OPT(\mathbf{v}) &\geq \sum_{i\in[n]} u_i(\mathbf{b}, v_i) \\ &\geq OPT(\mathbf{v}) - SW(\mathbf{b}, \mathbf{v})\end{aligned}$$

Rearranging, we get: $SW(\mathbf{b}, \mathbf{v}) \geq \frac{2}{3} OPT(\mathbf{v})$, as required. □

Bhawalkar and Roughgarden (2011) generalized the results from PNE to CCE and showed that in an S2PA with subadditive valuations the social welfare of every CCE satisfying strong NOB is at least $\frac{1}{2}$ of the optimal social welfare. Adding the assumption that the CCE satisfies also weak sNUB, it is straightforward to show (in a similar manner to the proof of Theorem 7.1) that:

Theorem 7.2. *In an S2PA with subadditive valuations, the social welfare of every CCE satisfying both strong NOB and weak sNUB, is at least $\frac{2}{3}$ of the optimal social welfare.*

Example 1.2 shows that the above bound is tight even for sNUB bid profiles.

Feldman et al. (2013) proved that in an S2PA with independent subadditive valuations the expected social welfare of every mixed Bayes Nash equilibrium which satisfies NOB, is at least $\frac{1}{4}$ of the optimal social welfare. Adding the weak sNUB assumption to the bid profile and using the fact the auction is $(1, 1)$-revenue guaranteed (as in the proof of Theorem 7.1) improves the lower bound on the BPoA to $\frac{1}{2}$ for independent subadditive valuations with both NOB and weak sNUB. Recall that the same lower bound is obtained without the NOB or equilibrium assumptions (See Theorems 4.1 and 3.4). We conclude that NOB does not improve the BPoA bound in this case.

Bhawalkar and Roughgarden (2011) gave an example of S2PA with subadditive valuations that does not have a PNE satisfying strong NOB. Fu et al. (2012) gave conditions on a valuation profile under which no conditional equilibrium exists, i.e., no S2PA PNE with NOB exists.

In general, a mixed equilibrium may not exist in infinite games (as in our case of continuous valuations and continuous bids). However, we can approximate a continuous auction with a finite discretized version which is guaranteed to admit a mixed equilibrium by Nash's theorem. We refer the readers to the relevant discussion in Feldman et al. (2013) and Cai and Papadimitriou (2014).

Under the finite discretized version of the auction, a mixed Bayes Nash equilibrium is guaranteed to exist. It remains to show, however, that the space of bid profiles satisfying both weak sNUB and NOB is non-empty.

Observation 7.1. Every S2PA with arbitrary monotone valuation functions admits a bid profile that satisfies both weak sNUB and NOB.

Proof. Let $S^*(\mathbf{v}) = (S_1^*(\mathbf{v}), \ldots, S_n^*(\mathbf{v}))$ be an optimal allocation. Consider the bid profile $\mathbf{b}$, where $b_{ij} = \frac{v_i(S_i^*(v))}{|S_i^*(v)|}$ for $j \in S_i^*(v)$ and 0 otherwise. Notice that each bidder i wins the items she gets in $S^*(\mathbf{v})$, i.e., $S_i(\mathbf{b}) = S_i^*(v)$. Hence, $\sum_{j\in S_i(\mathbf{b})} b_{ij} = \sum_{j\in S_i^*(\mathbf{v})} b_{ij} = v_i(S_i^*(v)) = v_i(S_i(\mathbf{b}))$, showing that $\mathbf{b}$ satisfies NOB. Moreover, as $S_i^*(\mathbf{v}) \backslash S_i(\mathbf{b}) = \emptyset$, $\mathbf{b}$ also satisfies weak sNUB. □

Finally, we show that for monotone valuations there always exists a (trivial) PNE bid profile that satisfies weak sNUB.

Proposition 7.1. *In S2PA with monotone valuations there always exists a PNE that satisfies weak sNUB.*

Proof. Let $S^*(\mathbf{v}) = (S_1^*(\mathbf{v}), \dots, S_n^*(\mathbf{v}))$ be a welfare maximizing allocation. Consider the bid profile $\mathbf{b}$ in which $b_{ij} = \infty$ if $j \in S_i^*$ and $b_{ij} = 0$ otherwise. One can easily verify that $\mathbf{b}$ is a PNE that satisfies weak sNUB. □

Declaration of competing interest

None.

Data availability

No data was used for the research described in the article.

Acknowledgments

We thank Noam Nisan for insightful comments regarding underbidding in the context of dominated strategies. We also thank Tomer Ezra for Example B.1 regarding XOS valuations and α-submodular valuations.

Appendix A. Missing proofs from Section 1.2

A.1. S2PA with XOS valuations and iNUB

Theorem A.1. *Every S2PA with XOS valuations and a bid profile* $\mathbf{b}$ *that satisfies weak iNUB is* $(1, m)$*-revenue guaranteed, where* m *is the number of items.*

Proof. We start with Inequality (7):

$$\sum_{i=1}^{n} \sum_{j \in S_i(\mathbf{b})} p_j(\mathbf{b}) \geq \sum_{i=1}^{n} \sum_{j \in S_i^*(\mathbf{v})} [v_i(S_i(\mathbf{b}) \cup \{j\}) - v_i(S_i(\mathbf{b})] \tag{A.1}$$

Consider the first term on the right hand side of Inequality (A.1). Let v_i^* be the maximizing additive valuation of player i with respect to her set $S_i^*(\mathbf{v})$. As v_i is an XOS function, for every set $S' \subseteq [m]$ we have $v_i(S') \geq \sum_{j \in S'} v_i^*(\{j\})$. Hence, together with monotonicity of v_i, we get:

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{j \in S_i^*(\mathbf{v})} v_i(S_i(\mathbf{b}) \cup \{j\}) &\geq \sum_{i=1}^{n} \sum_{j \in S_i^*(\mathbf{v})} v_i(\{j\}) \\
&\geq \sum_{i=1}^{n} \sum_{j \in S_i^*(\mathbf{v})} v_i^*(\{j\}) \\
&= \sum_{i=1}^{n} v_i(S_i^*(\mathbf{v})) \\
&= OPT(\mathbf{v})
\end{aligned} \tag{A.2}$$

Consider the second term on the right hand side of Inequality (A.1),

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{j \in S_i^*(\mathbf{v})} v_i(S_i(\mathbf{b})) &\leq \sum_{i=1}^{n} \sum_{j \in [m]} v_i(S_i(\mathbf{b})) \\
&= \sum_{j \in [m]} \sum_{i=1}^{n} v_i(S_i(\mathbf{b})) \\
&= \sum_{j \in [m]} SW(\mathbf{b}, \mathbf{v}) \\
&= m \cdot SW(\mathbf{b}, \mathbf{v})
\end{aligned} \tag{A.3}$$

Combining Equations (A.1), (A.2) and (A.3), we get:

$$\sum_{i=1}^{n} \sum_{j \in S_i(\mathbf{b})} p_j(\mathbf{b}) \geq OPT(\mathbf{v}) - m \cdot SW(\mathbf{b}, \mathbf{v}) \quad \square$$

Corollary A.1. *In an S2PA with XOS valuations, for every joint distribution $\mathcal{F} \in \Delta(\mathcal{V}_1 \times \ldots \times \mathcal{V}_n)$, possibly correlated, every mixed Bayes Nash equilibrium that satisfies weak iNUB has expected social welfare at least $\frac{1}{m+1}$ of the expected optimal social welfare.*

We next show an example of an instance with 4 items and 2 XOS bidders, where the PoA with NOB and iNUB is 1/2, which is no better than the guarantee obtained with NOB alone.

Example A.1. There are two XOS bidders, $\{1, 2\}$, and four items, $\{x, y, z, w\}$. Let,

$$v_1(S) = max\left\{a^1(S), a^2(S)\right\} \quad \text{and,}$$
$$v_2(S) = max\left\{a^3(S), a^4(S)\right\}$$

where:

$$a^1 = (a^1_x, a^1_y, a^1_z, a^1_w) = (2, 2, 0, 0),$$
$$a^2 = (a^2_x, a^2_y, a^2_z, a^2_w) = (0, 0, 1, 1),$$
$$a^3 = (a^3_x, a^3_y, a^3_z, a^3_w) = (0, 0, 2, 2),$$
$$a^4 = (a^4_x, a^4_y, a^4_z, a^4_w) = (1, 1, 0, 0).$$

The optimal allocation has welfare 8, giving items x and y to player 1 and items z and w to player 2. Consider the PNE bid profile $\mathbf{b} = (b_1, b_2)$, where:

$$b_1 = (b_{1x}, b_{1y}, b_{1z}, b_{1w}) = (0, 0, 1, 1) \quad \text{and,}$$
$$b_2 = (b_{2x}, b_{2y}, b_{2z}, b_{2w}) = (1, 1, 0, 0)$$

Player 1 wins items z and w and player 2 wins items x and y. $\mathbf{b}$ satisfies NOB and iNUB, and obtains welfare 4, which equals half of OPT.

A.2. S2PA with monotone valuations and iNUB

The following example shows that beyond subadditive valuations, the PoA can be arbitrarily bad under bid profiles satisfying iNUB.

Example A.2. 2 items: $\{x, y\}$, 2 single-minded bidders, who only derive value from the package of both items. Suppose $v_1(xy) = 1$, $v_2(xy) = R$ (where R is arbitrarily large), and the value for any strict subset of xy is 0. Consider the following bid profile (which is a PNE that adheres to iNUB): $b_{1x} = b_{1y} = R$, and $b_{2x} = b_{2y} = 0$. Under this bid profile, both items go to agent 1, for a PoA R.

Appendix B. Missing proofs from Section 2.2

B.1. XOS valuations are not α−SM

The following example shows that there exists an XOS function over identical items, sets $S \subset T$ and $j \notin T$ such that $v(j \mid S) < \alpha \cdot v(j \mid T)$ for every $\alpha > 0$.

Example B.1. Consider three identical items and valuation function v, where

$$v(S) = \begin{cases} 1, \text{ if } |S| \in \{1, 2\} \\ 1.5, \text{ if } |S| = 3 \end{cases}$$

One can verify that v is an XOS function. Let S be a set that contains a single item, and T be a set that contains two items, such that $S \subset T$. For $j \notin T$ it holds that $v(j \mid S) = 0$, and $v(j \mid T) = 0.5$. Thus, $v(j \mid S) < \alpha \cdot v(j \mid T)$ for every $\alpha > 0$.

B.2. α−SM valuations are not XOS

The following example shows that for every $\alpha \in (0, 1)$, there exists an α-SM function over identical items which is not XOS.

Example B.2. Consider a setting with a set M of three identical items, $0 < \alpha < 1$, and the following valuation function v:

$$v(S) = \begin{cases} 2, & \text{if } |S| = 1 \\ 2(1+\alpha), & \text{if } |S| = 2 \\ 2(2+\alpha), & \text{if } |S| = 3 \end{cases}$$

One can verify that v is α-SM. Let us assume by contradiction that v is XOS, i.e., there is a set $\mathcal{L}$ of additive valuations $\{a_\ell(\cdot)\}_{\ell \in \mathcal{L}}$, such that for every set $S \subseteq [m]$, $v(S) = max_{\ell \in \mathcal{L}} a_\ell(S)$. For every ℓ, it should hold that $a_\ell(S) \leq 2(1+\alpha)$ for every $|S| = 2$. It follows that for every ℓ, $a_\ell(M) \leq 3(1+\alpha) < 2(2+\alpha) = v(M)$, in contradiction to v being XOS.

Appendix C. Missing proofs from Section 2.3

Following is the proof of Theorem 2.1. The proof is based on the proofs in Roughgarden (2009) and Roughgarden et al. (2017) with the necessary adjustments.

Proof. (of Theorem 2.1) As the utility of each player is quasi-linear and payments are non negative,

$$\sum_{i \in [n]} \mathbb{E}_{\mathbf{b}} [u_i(\mathbf{b}, v_i)] \leq \mathbb{E}_{\mathbf{b}} [SW(\mathbf{b}, \mathbf{v})]$$

By smoothness, for each b in the support of $\mathbf{b}$, there exists a bid $b_i^*(\mathbf{v}) \in \mathcal{B}_i$ for each player i, s.t.:

$$\sum_{i \in [n]} u_i(b_i^*(\mathbf{v}), b_{-i}, v_i) \geq \lambda OPT(\mathbf{v}) - \mu SW(b, \mathbf{v})$$

Since $\mathbf{b}$ is a CCE, we get from Definition 2.4,

$$\mathbb{E}_{\mathbf{b}} [u_i(\mathbf{b}, v_i)] \geq \mathbb{E}_{\mathbf{b}} \left[u_i(b_i^*(\mathbf{v}), \mathbf{b}_{-i}, v_i)\right]$$

Putting it all together and using linearity of expectation we get,

$$\begin{aligned} \mathbb{E}_{\mathbf{b}} [SW(\mathbf{b}, \mathbf{v})] &\geq \sum_{i \in [n]} \mathbb{E}_{\mathbf{b}} [u_i(\mathbf{b}, v_i)] \\ &\geq \sum_{i \in [n]} \mathbb{E}_{\mathbf{b}} \left[u_i(b_i^*(\mathbf{v}), \mathbf{b}_{-i}, v_i)\right] \\ &\geq \mathbb{E}_{\mathbf{b}} [\lambda OPT(\mathbf{v}) - \mu SW(\mathbf{b}, \mathbf{v})] \\ &= \lambda OPT(\mathbf{v}) - \mu \mathbb{E}_{\mathbf{b}} [SW(\mathbf{b}, \mathbf{v})] \end{aligned} \tag{C.1}$$

Rearranging, we get: $\mathbb{E}_{\mathbf{b}} [SW(\mathbf{b}, \mathbf{v})] \geq \frac{\lambda}{1+\mu} OPT(\mathbf{v})$, as required. □

Following is the proof of Theorem 2.2. The proof is based on the proofs in Roughgarden (2012) and Syrgkanis and Tardos (2013) with the necessary adjustments.

Proof. (of Theorem 2.2) We give a proof for pure Bayes Nash equilibrium. The proof for mixed Bayes Nash equilibrium follows by adding in a straightforward way another expectation over the random actions chosen in the strategy profile σ.

Recall that the valuations v_i are independent and that the strategy $\sigma_i(v_i)$ of player i depends only on v_i. However, the smoothness definition (Definition 2.8) assumes that the hypothetical deviation of player i depends on the full valuation profile. Since player i doesn't know the valuation of the other players, she randomly samples an independent valuation profile $\mathbf{w} = (w_1, \ldots, w_n) \sim \mathcal{F}$ and decides on the hypothetical deviation $\mathbf{b}_i^*(v_i, \mathbf{w}_{-i})$ accordingly.

As σ is a BNE of the auction, using Inequality (2) we get,

$$\begin{aligned} \mathbb{E}_{\mathbf{v}} [u_i(\sigma(\mathbf{v}), v_i)] &\geq \mathbb{E}_{\mathbf{v}} \mathbb{E}_{\mathbf{w}} \left[u_i(\mathbf{b}_i^*(v_i, \mathbf{w}_{-i}), \sigma_{-i}(\mathbf{v}_{-i}), v_i)\right] \\ &= \mathbb{E}_{\mathbf{v}} \mathbb{E}_{\mathbf{w}} \left[u_i(\mathbf{b}_i^*(w_i, \mathbf{w}_{-i}), \sigma_{-i}(\mathbf{v}_{-i}), w_i)\right] \\ &= \mathbb{E}_{\mathbf{v}} \mathbb{E}_{\mathbf{w}} \left[u_i(\mathbf{b}_i^*(\mathbf{w}), \sigma_{-i}(\mathbf{v}_{-i}), w_i)\right] \end{aligned}$$

The equality follows by renaming due to independence. Now, let us sum over all players and then use the smoothness Inequality (3) and linearity of expectation to get,

$$\begin{aligned}\mathbb{E}_{\mathbf{v}}\left[\sum_{i\in[n]} u_i(\sigma(\mathbf{v}), v_i)\right] &\geq \mathbb{E}_{\mathbf{v}}\mathbb{E}_{\mathbf{w}}\left[\sum_{i\in[n]} u_i(\mathbf{b}_i^*(\mathbf{w}), \sigma_{-i}(\mathbf{v}_{-i}), w_i)\right] \\ &\geq \mathbb{E}_{\mathbf{v}}\mathbb{E}_{\mathbf{w}}[\lambda OPT(\mathbf{w}) - \mu SW(\sigma(\mathbf{v}), \mathbf{v})] \\ &= \lambda \mathbb{E}_{\mathbf{w}}[OPT(\mathbf{w})] - \mu \mathbb{E}_{\mathbf{v}}[SW(\sigma(\mathbf{v}), \mathbf{v})] \end{aligned} \tag{C.2}$$

As the utility of each player is quasi-linear and payments are non negative,

$$\mathbb{E}_{\mathbf{v}}\left[\sum_{i\in[n]} u_i(\sigma(\mathbf{v}), v_i)\right] \leq \mathbb{E}_{\mathbf{v}}[SW(\sigma(\mathbf{v}), \mathbf{v})]$$

Hence,

$$\mathbb{E}_{\mathbf{v}}[SW(\sigma(\mathbf{v}), \mathbf{v})] \geq \lambda \mathbb{E}_{\mathbf{v}}[OPT(\mathbf{v})] - \mu \mathbb{E}_{\mathbf{v}}[SW(\sigma(\mathbf{v}), \mathbf{v})]$$

The proof follows by rearranging. □

Appendix D. Missing proofs from Section 4

Following is an example of an S2PA with submodular bidders that admits a PNE satisfying weak sNUB but not weak iNUB.

Example D.1. Two submodular bidders: $\{1, 2\}$, and three items: $\{x, y, z\}$.

$$v_1(x) = 5, v_1(y) = 5, v_1(z) = 10, v_1(xy) = 10, v_1(xz) = 15, v_1(yz) = 15, v_1(xyz) = 16$$
$$v_2(x) = 8, v_2(y) = 8, v_2(z) = 15, v_2(xy) = 14, v_2(xz) = 15, v_2(yz) = 15, v_2(xyz) = 15$$

The optimal allocation is: $S_1^* = \{xy\}$, $S_2^* = \{z\}$, $OPT = 10 + 15 = 25$.
One can verify that the following bid profile $\mathbf{b}$ is a PNE:

$$b_{1x} = 3,\ b_{1y} = 3,\ b_{1z} = 8$$
$$b_{2x} = 8,\ b_{2y} = 8,\ b_{2z} = 2,$$

and the obtained allocation under $\mathbf{b}$ is:

$$S_1(b) = \{z\},\ S_2(b) = \{xy\},\ SW = 10 + 14 = 24.$$

We first show that $\mathbf{b}$ satisfies weak sNUB. Indeed,

$$6 = b_{1x} + b_{1y} \geq v_1(xy \mid z) = 6$$

and also,

$$2 = b_{2z} \geq v_2(z \mid xy) = 1.$$

However, $\mathbf{b}$ does not satisfy weak iNUB, since

$$3 = b_{1x} < v_1(x \mid z) = 5.$$

Appendix E. Missing proofs from Section 5

Following is the proof of Proposition 5.4.

Proof. Let $S^*(\mathbf{v}) = (S_1^*(\mathbf{v}), \ldots, S_n^*(\mathbf{v}))$ be a welfare maximizing allocation and assume w.l.o.g. that $|S_i^*(\mathbf{v})| \leq 1$ for every $i \in [n]$. Let j_i^*, be the item that player i is allocated in $S^*(\mathbf{v})$, i.e. $S_i^*(\mathbf{v}) = \left\{j_i^*\right\}$. If $S_i^*(\mathbf{v}) = \emptyset$, then $\left\{j_i^*\right\} = 0$ and for every $l \in [n]$ and every $T \in [m]$, $v_l(j_i^*) = 0$ and $v_l(\left\{j_i^*\right\} \cup T) = v_l(T)$. Consider the bid profile $\mathbf{b}$, in which every player bids its valuation on the item she wins, and bids the marginal value on the rest of the items, i.e. $b_{ij} = v_i(j_i^*)$ if $j = j_i^*$ and $b_{ij} = v_i(j \mid j_i^*)$ otherwise. We wish to show that $S^*(\mathbf{v})$ is a legal outcome of $\mathbf{b}$, and that $\mathbf{b}$ is a PNE which satisfies both NOB and sNUB.

To show that $S^*(\mathbf{v})$ is indeed a legal outcome of $\mathbf{b}$, we need to show that for every i, $b_{ij_i^*} \geq max_{l \neq i} b_{lj_i^*}$. Let $i' = argmax_{l \neq i} b_{lj_i^*}$. Assume by contradiction that $v_i(j_i^*) = b_{ij_i^*} < b_{i'j_i^*} = v_{i'}(j_i^* \mid j_{i'}^*)$. Then, as $v_{i'}(j_i^* \mid j_{i'}^*) > v_i(j_i^*) \geq 0$, and as

$v_{i'}$ is unit-demand, we have $v_{i'}(j_i^*, j_{i'}^*) = v_{i'}(j_i^*) = v_{i'}(j_{i'}^*) + v_{i'}(j_i^* \mid j_{i'}^*) > v_{i'}(j_{i'}^*) + v_i(j_i^*)$, in contradiction to the optimality of $S^*(\mathbf{v})$.

Given the allocation $S^*(\mathbf{v})$ and the fact the valuations are unit-demand, one can easily verify that **b** satisfies both NOB and sNUB. It is left to show that **b** is a PNE, i.e. no player can gain from unilateral deviation. As player utilities are unit-demand, we need to show that no player can gain from discarding the item she won, nor replacing it with a different item. We showed that for every i, $v_i(j_i^*) = b_{ij_i^*} \geq max_{l \neq i} b_{lj_i^*}$. Hence, the utility of a player under bid profile **b** is non negative, i.e. a player cannot gain from discarding the item she won.

It is left to show that no player can gain from replacing the item she won with a different item. Consider player i and let $i' = argmax_{l \neq i} b_{lj_i^*}$. We need to show that i cannot gain from replacing item j_i^* with: item $j_{i'}^*$; item $j_{i''}^*$, where $i'' \neq i$, $i'' \neq i'$ and $i'' \in S^*$; item $j \notin S^*$. From optimality of $S^*(\mathbf{v})$, $v_i(j_{i'}^*) + v_{i'}(j_i^*) \leq v_i(j_i^*) + v_{i'}(j_{i'}^*)$ and therefore, $u_i(j_{i'}^*) = v_i(j_{i'}^*) - v_{i'}(j_{i'}^*) \leq v_i(j_i^*) - v_{i'}(j_i^*) \leq v_i(j_i^*) - v_{i'}(j_i^* \mid j_{i'}^*) = u_i(j_i^*)$, where the last inequality follows from submodularity. Hence, player i cannot gain from replacing to item $j_{i'}^*$. Also, from optimality, $v_i(j_{i''}^*) + v_{i'}(j_i^*, j_{i'}^*) \leq v_i(j_i^*) + v_{i'}(j_{i'}^*) + v_{i''}(j_{i''}^*)$. Subtracting $v_{i'}(j_{i'}^*)$ from both sides gives, $v_i(j_{i''}^*) + v_{i'}(j_i^* \mid j_{i'}^*) \leq v_i(j_i^*) + v_{i''}(j_{i''}^*)$, i.e. $u_i(j_{i''}^*) = v_i(j_{i''}^*) - v_{i''}(j_{i''}^*) \leq v_i(j_i^*) - v_{i'}(j_i^* \mid j_{i'}^*) = u_i(j_i^*)$. That is, player i cannot gain from replacing to item $j_{i''}^*$. Now consider item $j \notin S^*$, i.e. item j is not allocated in the optimal allocation. From optimality of S^*, for every $i \in [n]$, $v_i(j) \leq v_i(j_i^*)$, and therefore, $b_{ij} = 0$. Also from optimality, $v_i(j_i^*) + v_{i'}(j_{i'}^*) \geq v_i(j) + v_{i'}(j_i^*, j_{i'}^*)$, i.e., $u_i(j_i^*) = v_i(j_i^*) - v_{i'}(j_i^* \mid j_{i'}^*) \geq v_i(j) = u_i(j)$. That is, player i cannot gain from replacing to an unallocated item j, which completes the proof. □

Following is the proof of Proposition 5.5.

Proof. Consider a setting with 3 items, $\{x, y, z\}$, and 2 players, $\{1, 2\}$, with the following submodular valuations: $v_1(x) = v_1(y) = 4$, $v_1(z) = 7.5$, $v_1(xy) = v_1(xz) = v_1(yz) = v_1(xyz) = 8$; $v_2(x) = v_2(y) = 3.5$, $v_2(z) = 5$, $v_2(xy) = 4$, $v_2(xz) = v_2(yz) = v_2(xyz) = 8$. We next show that no allocation is a PNE satisfying both NOB and sNUB.

- $S_1 = \emptyset$, $S_2 = \{x, y, z\}$. From sNUB, $b_{1x} \geq v_1(x) = 4$, $b_{1y} \geq v_1(y) = 4$, and $b_{1z} \geq v_1(z) = 7.5$, i.e. $b_{1x} + b_{1y} + b_{1z} \geq 4 + 4 + 7.5 = 15.5$. As $v_2(xyz) = 8$, the utility of player 2 is negative. A similar analysis applies for the case where $S_1 = \{x, y, z\}$, $S_2 = \emptyset$.
- $S_1 = \{x\}$, $S_2 = \{y, z\}$. From sNUB, $b_{1y} \geq v_1(y \mid x) = 4$. Therefore, $u_2(z) - u_2(yz) = v_2(z) - b_{1z} - (v_2(yz) - b_{1y} - b_{1z}) = v_2(z) - v_2(yz) + b_{1y} \geq 5 - 8 + 4 = 1$. Hence, for any bid profile satisfying sNUB, player 2 can gain from discarding item y. From symmetry, the same analysis applies for $S_1 = \{y\}$, $S_2 = \{x, z\}$.
- $S_1 = \{z\}$, $S_2 = \{x, y\}$. From NOB, $b_{2x} + b_{2y} \leq v_2(xy) = 4$. From sNUB, $b_{2z} \geq v_2(z \mid \{xy\}) = 8 - 4 = 4$. Therefore, $u_1(z) = v_1(z) - b_{2z} \leq 7.5 - 4 = 3.5$ and $u_1(xy) = v_1(xy) - (b_{2x} + b_{2y}) \geq 8 - 4 = 4$. Hence, player 1 can gain from replacing the item z with items $\{x, y\}$.
- $S_1 = \{x, y\}$, $S_2 = \{z\}$. From sNUB, $b_{2x} \geq v_2(x \mid z) = 3$ and $b_{2y} \geq v_2(y \mid z) = 3$. From NOB, $b_{2z} \leq v_2(z) = 5$. Therefore, $u_1(xy) = v_1(xy) - b_{2x} - b_{2y} \leq 8 - 3 - 3 = 2$ and $u_1(z) = v_1(z) - b_{2z} \geq 7.5 - 5 = 2.5$. Hence, player 1 can gain from replacing the items $\{x, y\}$ with item z.
- $S_1 = \{x, z\}$, $S_2 = \{y\}$. From sNUB, $b_{2z} \geq v_2(z \mid y) = 4.5$. Therefore, $u_1(x) - u_1(xz) = v_1(x) - b_{2x} - (v_1(xz) - b_{2x} - b_{2z}) = v_1(x) - v_1(xz) + b_{2z} \geq 4 - 8 + 4.5 = 0.5$. Hence, player 1 can gain from discarding item z. From symmetry, the same analysis applies for $S_1 = \{y, z\}$, $S_2 = \{x\}$. □

Appendix F. Revenue guaranteed auctions in settings with incomplete information

In the paper we defined both revenue guaranteed auctions and no-underbidding bid profile for full information with respect to a bid space $\mathcal{B}'$, which is the space of all bid profiles satisfying weak sNUB. We then used extension theorem to prove positive results for incomplete information on strategy space $\sigma : \mathcal{V}_1 \times \ldots \times \mathcal{V}_n \to \Delta(\mathcal{B}')$, i.e. for all strategies supported by bids $\sigma(\mathbf{v})$ satisfying weak sNUB. However, there might be strategies which satisfy no-underbidding only in expectation and yet guarantee lower bound on the expected revenue. Moreover, there might be auctions that are revenue guaranteed only in expectation. To prove results in incomplete information for such broader cases, we give here incomplete information definition to both revenue guaranteed auctions and no-underbidding strategy profiles.

Definition F.1 *(Revenue guaranteed auction of incomplete information).* An incomplete information auction is (γ, δ)-revenue guaranteed for some $0 \leq \gamma \leq \delta \leq 1$ with respect to a strategy space $\Sigma' \subseteq \Delta(\Sigma_1 \times \ldots \times \Sigma_n)$, if for every joint distribution $\mathcal{F} \in \Delta(\mathcal{V}_1 \times \ldots \times \mathcal{V}_n)$, possibly correlated, and for any strategy profile $\sigma \in \Sigma'$, the expected revenue of the auction is at least $\gamma\ \mathbb{E}_{\mathbf{v} \sim \mathcal{F}}[OPT(\mathbf{v})] - \delta\ \mathbb{E}_{\mathbf{v} \sim \mathcal{F}}\ \mathbb{E}_{\mathbf{b} \sim \sigma(\mathbf{v})}[SW(\mathbf{b}, \mathbf{v})]$.

Theorem F.1. *If an incomplete information auction is (γ, δ)-revenue guaranteed with respect to a strategy space $\Sigma' \subseteq \Delta(\Sigma_1 \times \ldots \times \Sigma_n)$, then for every joint distribution $\mathcal{F} \in \Delta(\mathcal{V}_1 \times \ldots \times \mathcal{V}_n)$, possibly correlated, and every strategy profile $\sigma \in \Sigma'$, in which the expected sum of players utility is non-negative, the expected social welfare is at least $\frac{\gamma}{1+\delta}$ of the expected optimal social welfare.*

Proof. As the utility of each player is quasi-linear and the expected sum of player utilities is non-negative, we use linearity of expectation and get,

$$\begin{aligned}
0 &\le \sum_{i\in[n]} \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\, \mathbb{E}_{\mathbf{b}\sim\sigma(\mathbf{v})}\,[u_i(\mathbf{b}, v_i)] \\
&= \sum_{i\in[n]} \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\, \mathbb{E}_{\mathbf{b}\sim\sigma(\mathbf{v})}\,[v_i(S_i(\mathbf{b}))] - \sum_{i\in[n]} \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\, \mathbb{E}_{\mathbf{b}\sim\sigma(\mathbf{v})}\,[P_i(\mathbf{b})] \\
&= \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\, \mathbb{E}_{\mathbf{b}\sim\sigma(\mathbf{v})}\,[SW(\mathbf{b},\mathbf{v})] - \sum_{i\in[n]} \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\, \mathbb{E}_{\mathbf{b}\sim\sigma(\mathbf{v})}\,[P_i(\mathbf{b})]
\end{aligned}$$

By the (γ,δ)-revenue guaranteed property for incomplete information,

$$\sum_{i\in[n]} \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\, \mathbb{E}_{\mathbf{b}\sim\sigma(\mathbf{v})}\,[P_i(\mathbf{b})] \ge \gamma\, \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\,[OPT(\mathbf{v})] - \delta\, \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\, \mathbb{E}_{\mathbf{b}\sim\sigma(\mathbf{v})}\,[SW(\mathbf{b},\mathbf{v})]$$

Putting it all together, we get,

$$\begin{aligned}
0 &\le \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\, \mathbb{E}_{\mathbf{b}\sim\sigma(\mathbf{v})}\,[SW(\mathbf{b},\mathbf{v})] - \sum_{i\in[n]} \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\, \mathbb{E}_{\mathbf{b}\sim\sigma(\mathbf{v})}\,[P_i(\mathbf{b})] \\
&\le \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\, \mathbb{E}_{\mathbf{b}\sim\sigma(\mathbf{v})}\,[SW(\mathbf{b},\mathbf{v})] - \gamma\, \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\,[OPT(\mathbf{v})] + \delta\, \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\, \mathbb{E}_{\mathbf{b}\sim\sigma(\mathbf{v})}\,[SW(\mathbf{b},\mathbf{v})] \\
&= (1+\delta)\, \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\, \mathbb{E}_{\mathbf{b}\sim\sigma(\mathbf{v})}\,[SW(\mathbf{b},\mathbf{v})] - \gamma\, \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\,[OPT(\mathbf{v})]
\end{aligned}$$

Rearranging, we get: $\mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\, \mathbb{E}_{\mathbf{b}\sim\sigma(\mathbf{v})}\,[SW(\mathbf{b},\mathbf{v})] \ge \frac{\gamma}{1+\delta}\, \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\,[OPT(\mathbf{v})]$, as required. □

Definition F.2 *(weak sNUB in expectation).* A strategy profile $\sigma \in \Delta(\Sigma_1 \times \ldots \times \Sigma_n)$ satisfies weak sNUB in expectation if for every joint distribution $\mathcal{F} \in \Delta(\mathcal{V}_1 \times \ldots \times \mathcal{V}_n)$, possibly correlated, and for every player i the following holds,

$$\mathbb{E}_{\mathbf{v}\sim\mathcal{F}|v_i}\, \mathbb{E}_{\mathbf{b}\sim\sigma(v)} \left[\sum_{j\in S_i^*(\mathbf{v})\setminus S_i(\mathbf{b})} b_{ij} \right] \ge \mathbb{E}_{\mathbf{v}\sim\mathcal{F}|v_i}\, \mathbb{E}_{\mathbf{b}\sim\sigma(v)} \left[v_i\left(S_i^*(\mathbf{v})\setminus S_i(\mathbf{b}) \mid S_i(\mathbf{b}) \right) \right]$$

Theorem F.2. *An incomplete information S2PA with monotone valuation functions is* $(1,1)$*-revenue guaranteed with respect to strategy profiles satisfying weak sNUB in expectation.*

Proof. We start with Lemma 2.1, use linearity of expectation and the fact that σ satisfies weak sNUB in expectation,

$$\begin{aligned}
\mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\, \mathbb{E}_{\mathbf{b}\sim\sigma(\mathbf{v})} \left[\sum_{i=1}^n \sum_{j\in S_i(\mathbf{b})} p_j(\mathbf{b}) \right] &\ge \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\, \mathbb{E}_{\mathbf{b}\sim\sigma(\mathbf{v})} \left[\sum_{i=1}^n \sum_{j\in S_i^*(\mathbf{v})\setminus S_i(\mathbf{b})} b_{ij} \right] \\
&= \sum_{i=1}^n \mathbb{E}_{\mathbf{v}\sim\mathcal{F}} \mathbb{E}_{\mathbf{b}\sim\sigma(\mathbf{v})} \left[\sum_{j\in S_i^*(\mathbf{v})\setminus S_i(\mathbf{b})} b_{ij} \right] \\
&\ge \sum_{i=1}^n \mathbb{E}_{\mathbf{v}\sim\mathcal{F}} \mathbb{E}_{\mathbf{b}\sim\sigma(v)} \left[v_i\left(S_i^*(\mathbf{v})\setminus S_i(\mathbf{b}) \mid S_i(\mathbf{b}) \right) \right] \\
&= \sum_{i=1}^n \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\, \mathbb{E}_{\mathbf{b}\sim\sigma(\mathbf{v})} \left[v_i(S_i(\mathbf{b}) \cup S_i^*(\mathbf{v})) - v_i(S_i(\mathbf{b})) \right] \\
&\ge \sum_{i=1}^n \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\, \mathbb{E}_{\mathbf{b}\sim\sigma(\mathbf{v})} \left[v_i(S_i^*(\mathbf{v})) - v_i(S_i(\mathbf{b})) \right] \\
&= \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\, \mathbb{E}_{\mathbf{b}\sim\sigma(\mathbf{v})} \left[\sum_{i=1}^n \left[v_i(S_i^*(\mathbf{v})) - v_i(S_i(\mathbf{b})) \right] \right] \\
&= \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\,[OPT(\mathbf{v})] - \mathbb{E}_{\mathbf{v}\sim\mathcal{F}}\, \mathbb{E}_{\mathbf{b}\sim\sigma(\mathbf{v})}\,[SW(\mathbf{b},\mathbf{v})]
\end{aligned}$$

The last Inequality follows from monotonicity of valuations. □

Corollary F.1. *In an incomplete information S2PA with monotone valuations, for every joint distribution $\mathcal{F} \in \Delta(\mathcal{V}_1 \times \ldots \times \mathcal{V}_n)$, possibly correlated, every mixed Bayes Nash equilibrium that satisfies weak sNUB in expectation has expected social welfare at least $\frac{1}{2}$ of the expected optimal social welfare.*

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