The Thirty-Fifth AAAI Conference on Artificial Intelligence (AAAI-21)

Simultaneous 2nd Price Item Auctions with No-Underbidding *

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Abstract

We study the price of anarchy (PoA) of simultaneous 2nd price auctions (S2PA) under a new natural condition of *no underbidding*, meaning that agents never bid on items less than their marginal values. We establish improved (mostly tight) bounds on the PoA of S2PA under no underbidding for different valuation classes (including unit-demand, submodular, XOS, subadditive, and general monotone valuations), in both full-information and incomplete information settings.

To derive our results, we introduce a new parameterized property of auctions, termed (γ, δ)-revenue guaranteed, which implies a PoA of at least $\gamma/(1+\delta)$. Via extension theorems, this guarantee extends to coarse correlated equilibria (CCE) in full information settings, and to Bayesian PoA (BPoA) in settings with incomplete information and arbitrary (correlated) distributions. We then show that S2PA are $(1, 1)$-revenue guaranteed with respect to bids satisfying no underbidding. This implies a PoA of at least $1/2$ for general monotone valuation, which extends to BPOA with arbitrary correlated distributions. Moreover, we show that (λ, μ)-smoothness combined with (γ, δ)-revenue guaranteed guarantees a PoA of at least $(\gamma + \lambda)/(1 + \delta + \mu)$. This implies a host of results, such as a tight PoA of $2/3$ for S2PA with submodular (or XOS) valuations, under no overbidding and no underbidding. Beyond establishing improved bounds for S2PA, the no underbidding assumption sheds new light on the performance of S2PA relative to simultaneous 1st price auctions.

1 Introduction

Simple auctions are often preferred in practice over complex truthful auctions. Starting with the seminal paper of Christodoulou, Kovács, and Schapira (2008), a lot of effort has been given to the study of *simultaneous item auctions*.

In simultaneous item auctions with n bidders and m items, every bidder i has a valuation function $v_i : 2^{[m]} \to \mathbb{R}^+$, where $v_i(S)$ is the value bidder i assigns to set $S \subseteq [m]$. Despite the combinatorial structure of the valuation, bidders submit bids on every item *separately and simultaneously*. In simultaneous first-price auctions (S1PA) every item is sold in a 1st-price auction; i.e., the highest bidder wins and pays her bid, whereas in simultaneous second-price auctions (S2PA) every item is sold in a 2nd-price auction; i.e., the highest bidder wins and pays the 2nd highest bid.

Clearly, these auctions are not truthful; bidders don't even have the language to express their true valuations. The performance of these auctions is often quantified by the *price of anarchy* (PoA), which measures their performance in equilibrium. Specifically, the PoA is defined as the ratio between the performance of an auction in its worst equilibrium and the performance of the optimal outcome. The price of anarchy in auctions has been of great interest to the AI community, see the influential survey of Roughgarden, Syrgkanis, and Tardos (2017), as well as recent work on learning dynamics in multi-unit auctions and games (Foster et al. 2016; Brânzei and Filos-Ratsikas 2019).

PoA and BPoA of S2PA: Background. The price of anarchy has been studied both in complete and incomplete information settings. In the former case, all valuations are known by all bidders. In the latter case, every bidder knows her own value and the probability distribution from which other bidder valuations are drawn. The common equilibrium notion in this case is *Bayes Nash equilibrium*, and the performance is quantified by the *Bayesian PoA* (BPoA) measure.

There are pathological examples showing that the PoA of S2PA can be arbitrarily bad, even in the simplest scenario of a single item auction with two bidders (Christodoulou, Kovács, and Schapira 2016). A common approach towards overcoming such pathological examples is the *no overbidding* (NOB) assumption, stating that the sum of player bids on the set of items she wins never exceeds its value. Consequently, all PoA results of S2PA use the NOB assumption.

The PoA and BPoA of simultaneous item auctions depend on the structure of the valuation functions. An important class is that of subadditive (SA) valuations, also known as *complement-free* valuations, where $v(S)+v(T) \geq v(S \cup T)$ for every sets of items S, T. A hierarchy of complement-free valuations is given in (Lehmann, Lehmann, and Nisan 2006), including unit-demand (UD), submodular (SM), XOS (XOS), and subadditive (SA) valuations, with the following strict containment relation: $\mathsf{UD} \subset \mathsf{SM} \subset \mathsf{XOS} \subset \mathsf{SA}$ (see Section 2.2 for formal definitions). Clearly, the PoA can only degrade as one moves to a larger valuation class. PoA and BPoA results under the no-overbidding as-

*For the full version, see https://arxiv.org/abs/2003.11857 (Feldman and Shabtai 2020). Due to space limitation, all missing statements and proofs are deferred to the full version.

sumption for the different classes have been obtained by Christodoulou, Kovács, and Schapira (2008) and follow-up work (Roughgarden 2009; Bhawalkar and Roughgarden 2011; Hassidim et al. 2011; Roughgarden 2012; Feldman et al. 2013; Syrgkanis and Tardos 2013; Christodoulou et al. 2016), and are summarized in Table 1. MON refers to the class of all monotone valuations.

1.1 No Underbidding (NUB)

Let $b_i = (b_{i1}, \ldots, b_{im})$ denote the bid vector of bidder i, where b_{ij} is the bid of bidder i for item j, and let $\mathbf{b} = (\mathbf{b_1}, \ldots, \mathbf{b_n})$ be the bid profile of all bidders. Consider the following example (taken from Christodoulou, Kovács, and Schapira (2016)), showing that the PoA for unit-demand (UD) valuations is at most $1/2$ (A valuation v is UD if there exist $v(1), \ldots, v(m)$, such that $v(S) = \max_{j\in S} v(j)$ for every set of items S).

Example 1.1. 2 bidders and 2 items, x, y. Bidder 1 is UD with values $v_1(x) = 2, v_1(y) = 1$. Bidder 2 is UD with values $v_2(x) = 1, v_2(y) = 2$. Consider the following bid profile, which is a pure Nash equilibrium (PNE) that adheres to NOB: $b_{1x} = b_{2y} = 0$, and $b_{1y} = b_{2x} = 1$. Under this bid profile, bidders 1 and 2 receive items y and x, respectively, for a social welfare of 2. The optimal welfare is 4.

Let us take a closer look at the Nash equilibrium in Example 1.1. In this equilibrium bidder 1 prefers item x, yet bids 0 on item x, and gets item y instead. Bidder 1's marginal value for item x, given her current allocation (item y), is $v_1(x \mid y) = v_1(xy) - v_1(y) = 1$. Given her current allocation y, bidding 0 on item x is *weakly dominated* by bidding 1 on x. Indeed, if bidder 1 receives item x, in addition to item y, her additional value is 1 and she pays at most 1. Therefore, it is only natural for her to bid at least her marginal value.

If a bidder bids on an item less than the item's marginal value, we say that she *underbids* (see Definition 4.1). In Example 1.1, bidder 1 underbids on item x. In Section 4 we show that underbidding in a 2nd price auction is *weakly dominated* in some precise technical sense.

In what sense is the outcome in Example 1.1 an equilibrium? While a Nash equilibrium is a descriptive, static notion, it is based on the underlying assumption that players engage in some dynamics, where they keep best responding to the current situation until a stable outcome is reached. In this dynamics, it is not likely that a player would bid on an item less than its marginal value. This is exactly what the no-underbidding assumption captures.

No-underbidding is not only a mere theoretical exercise. In second price auctions a lot of empirical evidence suggests that bidders tend to overbid, but not underbid (Kagel and Levin 1993; Harstad 2000; Cooper and Fang 2008; Roider and Schmitz 2012). It seems that "laboratory second-price auctions exhibit substantial and persistent overbidding, even with prior experience" (Harstad 2000). The no-underbidding assumption is also consistent with the assumption made by Nisan et al. (2011) that bidders break ties in favor of the highest bid that does not exceed their value. Yet, the POA literature employs no-overbidding as a standard assumption, and overlooked the no-underbidding phenomenon. The objective of this work is to better tie the theoretical work in this area to empirical evidence, by providing a theoretical foundation for the no-underbidding phenomenon.

Intuitively, no underbidding can improve welfare performance, as it drives item prices up, so that items become less attractive to low-value players. Consequently, bad equilibria, in which items are allocated to players with relatively low value, are excluded. A natural question is:

Main Question. *What is the performance (measured by PoA/BPoA) of simultaneous 2nd price item auctions under no underbidding?*

1.2 Our Contribution

We first introduce the notion of *item no underbidding* (iNUB), where no agent underbids on items (see Definition 4.4). One might think that by imposing both NOB and iNUB, the optimal welfare will be achieved. This is indeed the case for a single item auction (where the optimal welfare is achieved by imposing any one of these assumptions alone). However, even a simple scenario with 2 items and 2 unit-demand bidders can have a PNE with sub-optimal welfare. This is demonstrated in the following example.

Example 1.2. 2 bidders and 2 items, x, y. Bidder 1 is UD with values $v_1(x) = 3, v_1(y) = 2$. Bidder 2 is UD with values $v_2(x) = 2, v_2(y) = 3$. Consider the following PNE bid profile, which adheres to both NOB and iNUB: $b_{1x} = b_{2y} = 1, b_{1y} = b_{2x} = 2$. Under this bid profile, bidders 1 and 2 receive items y and x, respectively, for a social welfare of 4. The optimal welfare is 6. Thus, the PoA is $2/3$.

Submodular Valuations Our first result states that $2/3$ is the worst possible ratio for SM valuations and bid profiles satisfying both NOB and iNUB, even in settings with incomplete information (with a product distribution over valuations), see Corollary 5.5.

Beyond Submodular Valuations The above bounds do not carry over beyond submodular valuations. Consider first XOS valuations, defined as maximum over additive valuations. In the full version we show that the (B)PoA of S2PA with XOS valuations under iNUB is $\theta(\frac{1}{m})$. Moreover, for XOS valuations, iNUB may not provide any improvement over NOB alone. In particular, there exists an example where the PoA with NOB and iNUB is $\frac{1}{2}$, matching the guarantee provided by NOB alone (see full version).

To the best of our knowledge, this is the first PoA separation between SM and XOS valuations in simultaneous item auctions (or even between UD and XOS). Beyond SA valuations, the PoA can be arbitrarily bad under bid profiles satisfying iNUB (see full version).

To deal with valuations beyond SM, we consider a different no underbidding assumption, which applies to *sets of items*. For two sets S, T, the *marginal value* of T given S is defined as $v(T \mid S) = v(S \cup T) - v(S)$. A bidder is said to not underbid on a set S under bid profile $\mathbf{b}$ if $\sum_{j\in S} b_{ij} \geq v_i(S \mid S_i(\mathbf{b}))$. The new condition, *set no underbidding* (sNUB), imposes the set no underbidding condition on every

		UD / SM		XOS		SA		MON	
NOB	**PoA**	$\frac{1}{2}$	$\leftarrow$	$\frac{1}{2}$	[2016]	$\frac{1}{2}^{\dagger}$	[2011]	$O\left(\frac{1}{\sqrt{m}}\right)^{\sim}$	[2011; 2013]
	iBPoA	$\frac{1}{2}$	$\leftarrow$	$\frac{1}{2}$	[2016]	$\frac{1}{4}^{\sim}$	[2013]	$O\left(\frac{1}{\sqrt{m}}\right)^{\sim}$	[2011; 2013]
	BPoA	$O\left(\frac{1}{n^{1/4}}\right)^{\sim}$	[2011]	$O\left(\frac{1}{n^{1/4}}\right)^{\sim}$	[2011]	$O\left(\frac{1}{n^{1/4}}\right)^{\sim}$	[2011]	$O\left(\frac{1}{n^{1/4}}\right)^{\sim}$	[2011]

Table 1: Previous results for PoA of S2PA under the NOB assumption. PoA is the price of anarchy under full information; iBPoA and BPoA are the Bayesian PoA under independent valuation distributions and correlated valuation distributions, respectively. † This result assumes *strong* no-overbidding, i.e. the sum of player bids on *any* set of items never exceeds its value for that set. All results are tight, except those marked with $\sim$. Results derived as a special case of a more general result (to their right) are marked with $\leftarrow$.

bidder i with respect to the set $S = S_i^*(\mathbf{v}) \setminus S_i(\mathbf{b})$ (see Definition 4.5). With the sNUB definition, the $\frac{2}{3}$ PoA extends to SA valuations in full information settings (see full version), and to XOS valuations even in incomplete information settings (with product distributions), see Corollary 6.1.

For incomplete information we show that the BPoA of SA valuations is at least $\frac{1}{2}$ for *any* joint distribution (even correlated) and it can be obtained in a much stronger sense, namely for every bid profile with non-negative sum of utilities (even a non-equilibrium profile) satisfying sNUB. This also holds for markets with arbitrary monotone valuations (see full version).

The above results are summarized in Table 2.

Equilibrium existence PoA results make sense only when the corresponding equilibrium exists. We show that every market with XOS valuations admits a PNE satisfying sNUB and NOB. For SA valuations, a PNE satisfying NOB might not exist. However, under a finite discretized version of the auction, a mixed Bayes Nash equilibrium is guaranteed to exist, and we show that there is at least one bid profile that admits both sNUB and NOB with arbitrary monotone valuation functions.

S1PA vs. S2PA Interestingly, our results shed new light on the comparison between simultaneous 1st and 2nd price auctions. Table 3 specifies BPoA lower bounds for S1PA and S2PA under NOB, assuming independent valuation distributions. According to these results, one may conclude that S1PA perform better than their S2PA counterparts.

Our new results shed more light on the relative performance of S2PA and S1PA. When considering both no overbidding and no underbidding, the situation flips, and S2PA are superior to S1PA.[1] For XOS valuations, the $1 - 1/e$ bound for S1PA persists, but for S2PA the bound improves from $\frac{1}{2}(< 1 - \frac{1}{e})$ to $\frac{2}{3}(> 1 - \frac{1}{e})$. For SA valuations and independent valuation distributions, S2PA under sNUB performs as well as S1PA (achieving BPoA of $\frac{1}{2}$), however in S2PA the $\frac{1}{2}$ bound holds also for correlated valuation distributions. For valuations beyond SA, S2PA performs better ($\frac{1}{2}$ for S2PA and less than $\frac{1}{2}$ for S1PA).

[1]Note that no overbidding and no underbidding are not reasonable assumptions in 1st price auctions, where bidders pay their bids

2 Preliminaries

2.1 Auctions

Combinatorial auctions In a combinatorial auction a set of m non-identical items are sold to a group of n players. Let $\mathcal{S}_i$ be the set of possible allocations to player i, $\mathcal{V}_i$ the set of possible valuations of player i, and $\mathcal{B}_i$ the set of actions available to player i. Similarly, we let $\mathcal{S} \subseteq \mathcal{S}_1 \times \ldots \times \mathcal{S}_n$ be the allocation space of all players, $\mathcal{V} = \mathcal{V}_1 \times \ldots \times \mathcal{V}_n$ be the valuation space, and $\mathcal{B} = \mathcal{B}_1 \times \ldots \times \mathcal{B}_n$ be the action space. An allocation function maps an action profile to an allocation $\mathbf{S} = (S_1, \ldots, S_n) \in \mathcal{S}$, where S_i is the set of items allocated to player i. A payment function maps an action profile to a non negative payment $\mathbf{P} = (P_1, \ldots, P_n) \in \mathbb{R}_+$, where P_i is the payment of player i. We assume that the valuation function $v_i : \mathcal{S}_i \to \mathbb{R}^+$ of a player i, where $v_i \in \mathcal{V}_i$, is monotone and normalized, i.e., $\forall S \subseteq T \subseteq [m], v_i(S) \leq v_i(T)$ and also $v_i(\emptyset) = 0$. We let $\mathbf{v} = (v_1, \ldots, v_n)$ be the valuation profile. An outcome is a pair of allocation $\mathbf{S}$ and payment $\mathbf{P}$ and the revenue is the sum of all payments, i.e. $\mathcal{R}(\mathbf{b}) = \sum_{i \in [n]} P_i(\mathbf{b})$. We assume a quasi-linear utility function, i.e. $u_i(S_i, P_i, v_i) = v_i(S_i) - P_i$. We are interested in measuring the *social welfare*, which is the sum of bidder valuations, i.e., $SW(\mathbf{S}, \mathbf{v}) = \sum_{i \in [n]} v_i(S_i)$. Given a valuation profile $\mathbf{v}$, an optimal allocation is an allocation that maximizes the SW over all possible allocations. We denote by $OPT(\mathbf{v})$ the value of SW under optimal allocation.

Simultaneous item bidding auction. In a simultaneous item bidding auction (simultaneous item auction, in short) each item $j \in [m]$ is simultaneously sold in a separate auction. An action profile is a bid profile $\mathbf{b} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$, where $b_i = (b_{i1}, \ldots, b_{im})$ is an m-vector s.t. b_{ij} is the bid of player i for item j. The allocation of each item j is determined by the bids $(b_{1j}, \ldots, b_{nj})$. We use $S_i(\mathbf{b})$ to denote the items won by player i and $p_j(\mathbf{b})$ to denote the price paid for item j by the winner of item j. As allocation and payment are uniquely defined by the bid profile, we overload notation and write $u_i(\mathbf{b}, v_i)$ and $SW(\mathbf{b}, \mathbf{v})$.

In a simultaneous second price auction (S2PA), each item j is allocated to the highest bidder, who pays the second highest bid, i.e., $P_i = \sum_{j \in S_i(\mathbf{b})} \max_{k \neq i} b_{kj}$.

In a simultaneous first price auction (S1PA), each item j is allocated to the highest bidder, who pays her bid for that item, i.e., $P_i = \sum_{j \in S_i(\mathbf{b})} b_{ij}$.

Ties are broken arbitrarily but consistently.

		UD / SM	XOS	SA	MON
iNUB	**(B)PoA**	$\frac{1}{2}$*	$\Theta\left(\frac{1}{m}\right)$*		arbitrarily bad*
sNUB	**(B)PoA**	$\frac{1}{2}$ $\leftarrow$	$\frac{1}{2}$ $\leftarrow$	$\frac{1}{2}$ $\leftarrow$	$\frac{1}{2}$*
NOB+	**PoA**	$\frac{2}{3}$ $\leftarrow$	$\frac{2}{3}$*	$\frac{2}{3}^{\dagger}$*	$\frac{1}{2}$*
sNUB	**iBPoA**	$\frac{2}{3}$ $\leftarrow$	$\frac{2}{3}$*	$\frac{1}{2}$ $\leftarrow$	$\frac{1}{2}$*
	BPoA	$\frac{1}{2}$ $\leftarrow$	$\frac{1}{2}$ $\leftarrow$	$\frac{1}{2}$ $\leftarrow$	$\frac{1}{2}$*

Table 2: New results for PoA of S2PA. PoA is the price of anarchy under full information; iBPoA and BPoA are the Bayesian PoA under independent and correlated valuation distributions, respectively. † This result assumes *strong* no-overbidding, i.e. the sum of player bids on *any* set of items never exceeds its value for that set. All results are tight. Results derived in this paper are marked with *. Results derived as special cases of more general result (to their right) are marked with $\leftarrow$.

		UD / SM	XOS	SA	MON
S2PA	**NOB**	$\frac{1}{2}$ $\leftarrow$	$\frac{1}{2}$ [2016]	$\frac{1}{4}$ [2013]	$O\left(\frac{1}{\sqrt{m}}\right)$ [2011; 2013]
S1PA		$1-\frac{1}{e}$ $\leftarrow$	$1-\frac{1}{e}$ [2013; 2016]	$\frac{1}{2}$ [2013; 2016]	$\frac{1}{m}$ [2011]
S2PA	**sNUB+NOB**	$\frac{2}{3}$ $\leftarrow$	$\frac{2}{3}^*$	$\frac{1}{2}$ (corr) $\leftarrow$	$\frac{1}{2}^*$ (corr)

Table 3: BPoA results for S1PA and S2PA. (corr) refers to bounds that hold also for correlated distributions. Results derived in this paper are marked with *. Results derived as special cases of more general results (to their right) are marked with $\leftarrow$.

Full information setting: solution concepts and PoA. In the full information setting, the valuation profile $\mathbf{v} = (v_1, \ldots, v_n)$ is known to all players. The standard equilibrium concepts in this setting are pure Nash equilibrium (PNE), mixed Nash equilibrium (MNE), correlated Nash equilibrium (CE) and coarse correlated Nash equilibrium (CCE), where $PNE \subset MNE \subset CE \subset CCE$. Following are the definitions PNE and CCE, the definitions of MNE and CE are presented in the full version. As standard, for a vector $\mathbf{y}$, we denote by $\mathbf{y}_{-i}$ the vector $\mathbf{y}$ with the *i*th component removed, and by $\Delta(\Omega)$ the space of probability distributions over a finite set Ω.

Definition 2.1 (Pure Nash Equilibrium (PNE)). A bid profile $\mathbf{b} \in \mathcal{B}_1 \times \ldots \times \mathcal{B}_n$ is a PNE if for any $i \in [n]$ and for any $b_i^{'} \in \mathcal{B}_i$, $u_i(\mathbf{b}, v_i) \geq u_i(b_i^{'}, \mathbf{b}_{-i}, v_i)$.

Definition 2.2 (Coarse Correlated Nash Equilibriun (CCE)). A bid profile of randomized bids $\mathbf{b} \in \Delta(\mathcal{B}_1 \times \ldots \times \mathcal{B}_n)$ is a CCE if for any $i \in [n]$ and for any $b_i^{'} \in \mathcal{B}_i$, $\mathbb{E}_{\mathbf{b}}\left[u_i(\mathbf{b}, v_i)\right] \geq \mathbb{E}_{\mathbf{b}}\left[u_i(b_i^{'}, \mathbf{b}_{-i}, v_i)\right]$.

For a given instance of valuations $\mathbf{v}$, the price of anarchy (PoA) with respect to an equilibrium notion E is defined as: $PoA(\mathbf{v}) = \inf_{\mathbf{b} \in E} \frac{\mathbb{E}_{\mathbf{b}}[SW(\mathbf{b},\mathbf{v})]}{OPT(\mathbf{v})}$. For a family of valuations V, $PoA(\mathsf{V}) = \min_{\mathbf{v} \in \mathsf{V}} PoA(\mathbf{v})$.

Incomplete information setting: solution concepts and Bayesian PoA. In an incomplete information setting, player valuations are drawn from a commonly known, possibly correlated, joint distribution $\mathcal{F} \in \Delta(\mathcal{V}_1 \times \ldots \times \mathcal{V}_n)$, and the valuation v_i of each player is a private information which is known only to player i. The strategy of player i is a function $\sigma_i : \mathcal{V}_i \to \mathcal{B}_i$. Let Σ_i denote the strategy space of player i and $\Sigma = \Sigma_1 \times \ldots \times \Sigma_n$ the strategy space of all players. We denote by $\sigma(\mathbf{v}) = (\sigma_1(v_1), \ldots, (\sigma_n(v_n))$ the bid vector given a valuation profile $\mathbf{v}$.

In some cases, we assume that the joint distribution of the valuations is a product distribution, i.e., $\mathcal{F} = \mathcal{F}_1 \times \ldots \times \mathcal{F}_n \in \Delta(\mathcal{V}_1) \times \ldots \times \Delta(\mathcal{V}_n)$. In these cases, each valuation v_i is independently drawn from the commonly known distribution $\mathcal{F}_i \in \Delta(\mathcal{V}_i)$.

The standard equilibrium concepts in the incomplete information setting are the Bayes Nash equilibrium (BNE) and the mixed Bayes Nash equilibrium (MBNE):

Definition 2.3 (Bayes Nash Equilibriun (BNE)). A strategy profile σ is a BNE if for any $i \in [n]$, any $v_i \in \mathcal{V}_i$ and any $b_i^{'} \in \mathcal{B}_i$, $\mathbb{E}_{\mathbf{v}_{-i}|v_i}[u_i(\sigma_i(v_i), \sigma_{-i}(\mathbf{v}_{-i}), v_i)] \geq \mathbb{E}_{\mathbf{v}_{-i}|v_i}[u_i(b_i^{'}, \sigma_{-i}(\mathbf{v}_{-i}), v_i)]$

Note that if player valuations are independent, we can omit the conditioning on v_i in Definition 2.3. The definition for MBNE is similar to BNE, but with another expectation over the strategy profile σ (see definition in the full version). The Bayes Nash price of anarchy is:

$$BPoA = \inf_{\mathcal{F},\, \sigma \in BNE} \frac{\mathbb{E}_{\mathbf{v}}\left[SW(\sigma(\mathbf{v}), \mathbf{v})\right]}{\mathbb{E}_{\mathbf{v}}\left[OPT(\mathbf{v})\right]}$$

The mixed BPoA is defined similarly w.r.t. MBNE.

2.2 Valuation Classes

Hereinafter we present the valuation functions considered in this paper. As standard, for a valuation v, item j and set S, we denote the marginal value of j, given S, as $v(j|S)$; i.e., $v(j|S) = v(S \cup \{j\}) - v(S)$. Similarly, the marginal value of a set $S^{'}$, given a set S, is $v(S^{'}|S) = v(S \cup S^{'}) - v(S)$. Following are the valuation classes we consider:

- Unit-demand (UD): There exist values $v_1, \ldots, v_m$ s.t. $v(S) = max_{j \in S}\, v_j \;\; \forall S \subseteq [m]$.

- Submodular (SM): $\forall S \subseteq T \subseteq [m]$ and $j \notin T$, $v(j \mid S) \geq v(j \mid T)$.
- XOS (fractionally subadditive): There exists a set $\mathcal{L}$ of additive valuations $\{a_\ell(\cdot)\}_{\ell \in \mathcal{L}}$, s.t. $\forall S \subseteq [m]$, $v(S) = max_{\ell \in \mathcal{L}}\, a_\ell(S)$.
- Subadditive (SA): $\forall S, T \subseteq [m], v(S)+v(T) \geq v(S \cup T)$.
- Monotone (MON): $\forall S \subseteq T \subseteq [m], v(S) \leq v(T)$.

A strict containment hierarchy of the above valuation classes is known: $UD \subset SM \subset XOS \subset SA \subset MON$.

Lemma 2.4. *For any SM function v and any sets S, S': $\sum_{j \in S'} v(j \mid S) \geq v(S' \mid S)$.*

In the full version we generalize the SM class to $\alpha-$SM, where the parameter α denotes "how far" the valuation is from a SM valuation, and generalize our results to $\alpha-$SM.

2.3 Smooth Auctions

We use the following smoothness definition, based on Roughgarden (2009); Roughgarden, Syrgkanis, and Tardos (2017):

Definition 2.5 (**Smooth auction**)**.** An auction is $(\lambda, \mu)-$smooth for parameters $\lambda, \mu \geq 0$ with respect to a bid space $\mathcal{B}' \subseteq \mathcal{B}_1 \times \ldots \times \mathcal{B}_n$, if for any valuation profile $\mathbf{v} \in \mathcal{V}_1 \times \ldots \times \mathcal{V}_n$ and any bid profile $\mathbf{b} \in \mathcal{B}'$ there exists a bid $b_i^*(\mathbf{v}) \in \mathcal{B}_i$ for each player i, s.t.:

$$\sum_{i \in [n]} u_i(b_i^*(\mathbf{v}), \mathbf{b}_{-i}, v_i) \geq \lambda \cdot OPT(\mathbf{v}) - \mu \cdot SW(\mathbf{b}, \mathbf{v}) \quad (1)$$

It is shown in Roughgarden (2009); Roughgarden, Syrgkanis, and Tardos (2017) that for every $(\lambda, \mu)-$smooth auction, the social welfare of any pure NE is at least $\frac{\lambda}{1+\mu}$. Via extension theorems, this bound extends to CCE in full-information settings and to Bayes NE in settings with incomplete information (under independent distribution of valuations).

A standard assumption in essentially all previous work on the PoA of S2PA (Christodoulou, Kovács, and Schapira 2016; Feldman et al. 2013; Roughgarden 2012, 2009; Bhawalkar and Roughgarden 2011) is *no overbidding*, meaning that players do not overbid on items they win. I.e.,

Definition 2.6 (**No overbidding (NOB)**)**.** Given a valuation profile $\mathbf{v} \in \mathcal{V}_1 \times \ldots \times \mathcal{V}_n$, a bid profile $\mathbf{b} \in \mathcal{B}$ is said to satisfy NOB if for every player i it holds that $\sum_{j \in S_i(\mathbf{b})} b_{ij} \leq v_i(S_i(\mathbf{b}))$.

Theorem 2.7. ***(based on (Christodoulou, Kovács, and Schapira 2016; Roughgarden 2009))**: S2PA with XOS valuations is $(1,1)-$smooth, w.r.t. bid profiles satisfying NOB.*

Theorem 2.7 implies a lower bound of $\frac{1}{2}$ on the Bayesian PoA of S2PA with XOS valuations. This result is tight, even with respect to UD valuations in full information settings (Christodoulou, Kovács, and Schapira 2016).

3 Revenue Guaranteed Auctions

In this section we define a new parameterized notion called *revenue guaranteed* and infer results for the PoA of revenue guaranteed auctions in full information setting and pure NE. Similarly to the smoothness framework, we augment our results with two extension theorems, one for PoA with respect to CCE, and one for BPoA in settings with incomplete information. Moreover, the BPoA result holds also in cases where the joint distribution of bidder valuations is correlated (whereas previous BPoA results hold only under a product distribution over valuations). Combining the two tools of smoothness and revenue guaranteed, we get an improved bound.

Definition 3.1 (**Revenue guaranteed auction**)**.** An auction is $(\gamma, \delta)-$revenue guaranteed for some $0 \leq \gamma \leq \delta \leq 1$ with respect to a bid space $\mathcal{B}' \subseteq \mathcal{B}_1 \times \ldots \times \mathcal{B}_n$, if for any valuation profile $\mathbf{v} \in \mathcal{V}_1 \times \ldots \times \mathcal{V}_n$ and for any bid profile $\mathbf{b} \in \mathcal{B}'$ the revenue of the auction is at least $\gamma \cdot OPT(\mathbf{v}) - \delta \cdot SW(\mathbf{b}, \mathbf{v})$.

3.1 Full Information

The following theorem establishes welfare guarantees on every pure bid profile of a $(\gamma, \delta)-$revenue guaranteed auction in which the sum of player utilities is non-negative.

Theorem 3.2. *If an auction is $(\gamma, \delta)-$revenue guaranteed w.r.t. a bid space $\mathcal{B}' \subseteq \mathcal{B}_1 \times \cdots \times \mathcal{B}_n$, then for any pure bid profile $\mathbf{b} \in \mathcal{B}'$, in which the sum of player utilities is non-negative, the SW is at least $\frac{\gamma}{1+\delta}$ of the optimal SW.*

Proof. Using quasi-linear utilities and non-negative sum of player utilities, $0 \leq \sum_{i \in [n]} u_i(\mathbf{b}, v_i) = \sum_{i \in [n]} v_i(S_i(\mathbf{b})) - \sum_{i \in [n]} P_i(\mathbf{b}) = SW(\mathbf{b}, \mathbf{v}) - \sum_{i \in [n]} P_i(\mathbf{b})$. By the $(\gamma, \delta)-$revenue guaranteed property, $\sum_{i \in [n]} P_i(b) \geq \gamma OPT(\mathbf{v}) - \delta SW(b, \mathbf{v})$. Punting it all together, we get:

$$0 \leq SW(\mathbf{b}, \mathbf{v}) - \sum_{i \in [n]} P_i(\mathbf{b}) \leq (1+\delta) SW(\mathbf{b}, \mathbf{v}) - \gamma OPT(\mathbf{v}). \quad (2)$$

Rearranging gives: $SW(\mathbf{b}, \mathbf{v}) \geq \frac{\gamma}{1+\delta} OPT(\mathbf{v})$, as required. □

Definition 3.1 considers pure bid profiles, but Theorem 3.2 applies to the more general setting of randomized bid profiles, possibly correlated, as cast in the following extension theorem.

Theorem 3.3. *If an auction is $(\gamma, \delta)-$revenue guaranteed with respect to a bid space $\mathcal{B}' \subseteq \mathcal{B}_1 \times \ldots \times \mathcal{B}_n$, then for any bid profile $\mathbf{b} \in \Delta(\mathcal{B}')$, in which the sum of the expected utilities of the players is non-negative, the expected social welfare is at least $\frac{\gamma}{1+\delta}$ of the optimal social welfare.*

The proof is identical to the proof of Theorem 3.2, except adding expectation over $\mathbf{b}$ to every term, using the fact the the auction is $(\gamma, \delta)-$revenue guaranteed for every b in the support of $\mathbf{b}$, and using linearity of expectation.

Clearly, in every equilibrium (including CCE) the expected utility of every player is non-negative. It therefore follows that the expected welfare in any CCE is at least $\frac{\gamma}{1+\delta}$ of the optimal social welfare.

For an auction that is both smooth and revenue guaranteed, we give a better bound on the price of anarchy:

Theorem 3.4. *If an auction is $(\lambda, \mu)-$smooth with respect to a bid space $\mathcal{B}'$ and $(\gamma, \delta)-$revenue guaranteed with respect to a bid space $\mathcal{B}$", then the expected social welfare at any CCE $\in \Delta(\mathcal{B}' \cap \mathcal{B}")$ of the auction is at least $\frac{\lambda+\gamma}{1+\mu+\delta}$ of the optimal social welfare.*

Proof. Let $\mathbf{b} \in \Delta(\mathcal{B}' \cap \mathcal{B}")$ be a CCE of the auction. Since the auction is $(\lambda, \mu)-$smooth with respect to a bid space $\mathcal{B}'$ (Roughgarden 2009):

$$\sum_{i \in [n]} \mathbb{E}_{\mathbf{b}}\left[u_i(\mathbf{b}, v_i)\right] \geq \lambda \cdot OPT(\mathbf{v}) - \mu \cdot \mathbb{E}_{\mathbf{b}}\left[SW(\mathbf{b}, \mathbf{v})\right]$$

From Equation (2) we get,

$$\mathbb{E}_{\mathbf{b}}\left[SW(\mathbf{b}, \mathbf{v})\right] - \sum_{i \in [n]} \mathbb{E}_{\mathbf{b}}\left[P_i(\mathbf{b})\right] \leq$$
$$(1+\delta) \cdot \mathbb{E}_{\mathbf{b}}\left[SW(\mathbf{b}, \mathbf{v})\right] - \gamma \cdot OPT(\mathbf{v})$$

As utilities are quasi-linear, the left hand side of the above two inequalities are equal. Rearranging, we get: $\mathbb{E}_{\mathbf{b}}\left[SW(\mathbf{b}, \mathbf{v})\right] \geq \frac{\lambda+\gamma}{1+\mu+\delta} OPT(\mathbf{v})$, as required. □

3.2 Incomplete Information: Extension Theorem

Following is an extension theorem for settings with incomplete information.

Theorem 3.5. *If an auction is $(\gamma, \delta)-$revenue guaranteed with respect to a bid space $\mathcal{B}' \subseteq \mathcal{B}_1 \times \cdots \times \mathcal{B}_n$, then for every joint distribution $\mathcal{F} \in \Delta(\mathcal{V}_1 \times \ldots \times \mathcal{V}_n)$, possibly correlated, and every strategy profile $\sigma : \mathcal{V}_1 \times \ldots \times \mathcal{V}_n \to \Delta(\mathcal{B}')$, in which the expected sum of player utilities is non-negative, the expected social welfare is at least $\frac{\gamma}{1+\delta}$ of the expected optimal social welfare.*

As the expected utility of each player is non-negative at any equilibrium strategy profile, we infer that if an auction is $(\gamma, \delta)-$revenue guaranteed w.r.t. a bid space $\mathcal{B}'$, then for every joint distribution $\mathcal{F} \in \Delta(\mathcal{V}_1 \times \ldots \times \mathcal{V}_n)$, possibly correlated, the expected SW at any MBNE, $\sigma : \mathcal{V}_1 \times \ldots \times \mathcal{V}_n \to \Delta(\mathcal{B}')$, is at least $\frac{\gamma}{1+\delta}$ of the expected optimal SW.

For an auction that is both smooth and revenue guaranteed, we give a better bound on the price of anarchy, if the joint distribution $\mathcal{F}$ is a product distribution:

Theorem 3.6. *If an auction is $(\lambda, \mu)-$smooth with respect to a bid space $\mathcal{B}'$ and $(\gamma, \delta)-$revenue guaranteed with respect to a bid space $\mathcal{B}$", then for every product distribution $\mathcal{F}$, every mixed Bayes Nash equilibrium, $\sigma : \mathcal{V}_1 \times \ldots \times \mathcal{V}_n \to \Delta(\mathcal{B}' \cap \mathcal{B}")$, has expected social welfare at least $\frac{\lambda+\gamma}{1+\mu+\delta}$ of the expected optimal social welfare.*

Remark. In the full version we give an incomplete information definition of revenue guaranteed auctions, which allows us to get positive results for auctions that are revenue guaranteed only in expectation.

4 S2PA with No-Underbidding

We first define what it means to *underbid* on an item. Let $\mathbf{b}_{-j}$ denote the bids of all bidders on items $[m] \setminus \{j\}$.

Definition 4.1 (item underbidding). Fix $\mathbf{b}_{-j}$. Player i is said to *underbid* on item j if: $b_{ij} < v_i(j \mid S_i(\mathbf{b}_{-j}))$, where $S_i(\mathbf{b}_{-j}) = \{k \mid k \neq j, b_{ik} = \max_l \{b_{lk}\}\}$.

That is, we say that player i underbids on item j in a bid profile $\mathbf{b}$ if i's bid on item j is smaller than the marginal valuation of j w.r.t the set of items other than j won by i.

We next show that underbidding is weakly dominated in a precise sense that we define next. Consider a bid profile $\mathbf{b}$. Let $\mathbf{b}_{-j}$ be the bids of all bidders on all items except j, and let b_{-ij} be the bids on item j of all players, except player i.

Definition 4.2 (weakly dominated). A bid b'_{ij} is *weakly dominated* by bid b_{ij}, with respect to $\mathbf{b}_{-j}$, if the following two conditions hold:

1. $u_i(b_{ij}, b_{-ij}, \mathbf{b}_{-j}; v_i) \geq u_i(b'_{ij}, b_{-ij}, \mathbf{b}_{-j}; v_i), \forall b_{-ij}$
2. There exists b_{-ij} such that inequality (1) holds strictly.

The following lemma shows that underbidding on an item is weakly dominated by bidding its marginal value.

Lemma 4.3. *In S2PA, for every player i, every item j, and every bid profile $\mathbf{b}_{-j}$, underbidding on item j is weakly dominated by bidding $b_{ij} = v_i(j \mid S_i(b_{-j}))$, with respect to $\mathbf{b}_{-j}$.*

Motivated by the above lemma, we next define the notion of item no underbidding (iNUB):

Definition 4.4 (Item No-UnderBidding (iNUB)). Given a valuation profile $\mathbf{v} \in \mathcal{V}_1 \times \ldots \times \mathcal{V}_n$, a bid profile $\mathbf{b} \in \mathcal{B}$ satisfies iNUB if there exists a welfare maximizing allocation, $S^*(\mathbf{v})$, such that for every player i and every item $j \in S^*_i(\mathbf{v}) \setminus S_i(\mathbf{b})$ it holds that: $b_{ij} \geq v_i(j \mid S_i(\mathbf{b}))$.

We also define the notion of set no underbidding (sNUB):

Definition 4.5 (Set No underbidding (sNUB)). Given a valuation profile $\mathbf{v} \in \mathcal{V}_1 \times \ldots \times \mathcal{V}_n$, we say that a bid profile $\mathbf{b} \in \mathcal{B}$ satisfies sNUB if there exists a welfare maximizing allocation, $S^*(\mathbf{v})$, such that for every player i, it holds that $\sum_{j \in S'} b_{ij} \geq v_i(S' \mid S_i(\mathbf{b}))$, where $S' = S^*_i(\mathbf{v}) \setminus S_i(\mathbf{b})$.

In Section 5 we show that if valuations are submodular, every bid profile that satisfies iNUB, also satisfies sNUB. The opposite is not necessarily true, as demonstrated in the full version. For unit-demand bidders, iNUB and sNUB coincide (as one can assume w.l.o.g. that every bidder receives a single item in an optimal allocation).

Lemma 4.6. *Consider an S2PA and a valuation $\mathbf{v}$. Let $S^*(\mathbf{v}) = (S^*_1(\mathbf{v}), \ldots, S^*_n(\mathbf{v}))$ be a welfare-maximizing allocation. Then, for every bid profile $\mathbf{b}$ the following holds:*

$$\sum_{i=1}^{n} \sum_{j \in S_i(\mathbf{b})} p_j(\mathbf{b}) \quad \geq \quad \sum_{i=1}^{n} \sum_{j \in S^*_i(\mathbf{v}) \setminus S_i(\mathbf{b})} b_{ij}$$

The following shows that sNUB is a powerful property.

Theorem 4.7. *S2PA is $(1, 1)-$revenue guaranteed w.r.t. bid profiles satisfying sNUB for any* MON *valuations.*

Proof. In what follows, the first inequality follows by Lemma 4.6, the second inequality follows by sNUB, and the last inequality follows by monotonicity of valuations:

$$\sum_{i=1}^{n}\sum_{j\in S_i(\mathbf{b})} p_j(\mathbf{b}) \geq \sum_{i=1}^{n}\sum_{j\in S_i^*(\mathbf{v})\setminus S_i(\mathbf{b})} b_{ij} \geq \sum_{i=1}^{n} [\, v_i(\, S_i(\mathbf{b}) \cup (S_i^*(\mathbf{v})\setminus S_i(\mathbf{b}))\,) - v_i(S_i(\mathbf{b}))\,] = \sum_{i=1}^{n} [\, v_i(\, S_i(\mathbf{b}) \cup S_i^*(\mathbf{v})\,) - v_i(S_i(\mathbf{b}))\,] \geq \sum_{i=1}^{n} [\, v_i(S_i^*(\mathbf{v})) - v_i(S_i(\mathbf{b}))\,] = OPT(\mathbf{v}) - SW(\mathbf{b},\mathbf{v})$$

□

The following follows from Theorems 4.7 and 3.5.

Corollary 4.8. *In an S2PA with monotone valuations, for every joint distribution $\mathcal{F} \in \Delta(\mathcal{V}_1 \times \ldots \times \mathcal{V}_n)$, possibly correlated, every mixed Bayes Nash equilibrium that satisfies sNUB has expected social welfare at least $\frac{1}{2}$ of the expected optimal social welfare.*

Remark. In the full version we give incomplete information definition of no-underbidding strategy profiles, which allows us to get positive results for a wider strategy space.

5 S2PA with Submodular (SM) Valuations

In this section we study S2PA with SM valuations. We first show that for this class of valuations, the notion of iNUB suffices for establishing positive results.

Theorem 5.1. *Every S2PA with SM valuations is $(1,1)-$revenue guaranteed w.r.t. bids satisfying iNUB.*

An immediate corollary from Theorems 3.5 and 5.1 is:

Corollary 5.2. *In an S2PA with SM valuations, for every joint distribution $\mathcal{F} \in \Delta(\mathcal{V}_1\times\ldots\times\mathcal{V}_n)$, possibly correlated, and every strategy profile σ that satisfies iNUB for which the expected sum of player utilities is non-negative, the expected social welfare is at least $\frac{1}{2}$ of the expected optimal social welfare.*

The $\frac{1}{2}$ bound is tight, even with respect to unit-demand valuations and even in equilibrium, as shown in the following proposition.

Proposition 5.3. *There exists an S2PA with UD valuations that admits a PNE satisfying iNUB, where the social welfare in equilibrium is $\frac{1}{2}$ of the optimal social welfare.*

Proof. Consider an S2PA with two unit demand players and 2 items, $\{x, y\}$, where $v_1(x) = 2$, $v_1(y) = 1$, $v_2(x) = 1$ and $v_2(y) = 2$. An optimal allocation gives item x to player 1 and item y to player 2, for a welfare of 4. Consider the following bid profile $\mathbf{b}$: $b_{1x} = 1$, $b_{1y} = 100$, $b_{2x} = 100$ and $b_{2y} = 1$. Player 1 wins item y for a price of 1, and player 2 wins item x for a price of 1. It is easy to see that $\mathbf{b}$ is a PNE that satisfies iNUB. The social welfare of this equilibrium is 2, which is $\frac{1}{2}$ of the optimal social welfare. □

For SM valuations, iNUB implies sNUB:

Proposition 5.4. *For every SM valuation $\mathbf{v}$, every bid profile $\mathbf{b}$ that satisfies iNUB also satisfies sNUB.*

Proof. By iNUB, for every item $j \in S_i^*(\mathbf{v}) \setminus S_i(\mathbf{b})$, it holds that $b_{ij} \geq v_i(j \mid S_i(\mathbf{b}))$. It follows that $\sum_{j\in S_i^*(\mathbf{v})\setminus S_i(\mathbf{b})} b_{ij} \geq \sum_{j\in S_i^*(\mathbf{v})\setminus S_i(\mathbf{b})} v_i(j \mid S_i(\mathbf{b})) \geq v_i(S_i^*(\mathbf{v}) \setminus S_i(\mathbf{b}) \mid S_i(\mathbf{b}))$. The last inequality follows by Lemma 2.4. □

Therefore, Theorem 4.7 and Corollary 4.8 also apply to every SM valuation that satisfies iNUB.

For profiles satisfying both iNUB and NOB, we get a better bound as a corollary from Theorems 5.1, 2.7, and 3.6.

Corollary 5.5. *In an S2PA with SM valuations, for every product distribution $\mathcal{F}$, every mixed Bayes Nash equilibrium that satisfies both NOB and iNUB has expected social welfare at least $\frac{2}{3}$ of the expected optimal social welfare. This result is tight.*

The bound of $\frac{2}{3}$ for SM valuations is tight even with respect to a PNE with UD valuations (see Example 1.2).

In the next section, we show that every S2PA with XOS valuations admits a PNE that satisfies both NOB and sNUB (Theorem 6.2). Since every submodular valuation is XOS, the existence result applies also to submodular valuations.

6 S2PA with XOS Valuations

XOS Valuations under iNUB: For XOS valuations, iNUB does not imply sNUB, thus iNUB does not lead automatically to good PoA. An example with PoA of $\frac{2}{m}$ is given in the full version.

XOS Valuations under sNUB: As Theorem 4.7 and Corollary 4.8 apply for MON valuation functions, the PoA is at least $\frac{1}{2}$ with respect to bid profiles satisfying sNUB. An immediate corollary from Theorems 2.7, 4.7 and 3.6 is:

Corollary 6.1. *In an S2PA with XOS valuations, for every product distribution $\mathcal{F}$, every mixed Bayes Nash equilibrium that satisfies both NOB and sNUB has expected social welfare at least $\frac{2}{3}$ of the expected optimal social welfare.*

This result is tight (see Example 1.2). We next show that every S2PA with XOS valuations admits a PNE that satisfies both NOB and sNUB.

Theorem 6.2. *In S2PA with XOS valuations there always exists at least one PNE that satisfies both NOB and sNUB.*

Proof. Christodoulou, Kovács, and Schapira (2016) showed that every S2PA with XOS valuations admits a PNE satisfying NOB. We show that the same PNE satisfies sNUB as well. Let $S^*(\mathbf{v}) = (S_1^*(\mathbf{v}), \ldots, S_n^*(\mathbf{v}))$ be a welfare maximizing allocation, and let a_i^* be an additive valuation such that $v_i(S_i^*(\mathbf{v})) = \sum_{j\in S_i^*(\mathbf{v})} a_{ij}^*$. Consider the bid profile in which every player bids according to the maximizing additive valuation with respect to her set $S_i^*(\mathbf{v})$, i.e., $b_{ij} = a_{ij}^*$ for every $j \in S_i^*(\mathbf{v})$ and $b_{ij} = 0$ otherwise. One can easily verify that this bid profile is a PNE that satisfies NOB. It thus remains to show that it also satisfies sNUB. Recall that sNUB imposes restrictions on the bid values of the set $S' = S_i^*(\mathbf{v})\setminus S_i(\mathbf{b})$. Under the above bid profile we have $S_i(\mathbf{b}) = S_i^*(\mathbf{v})$, i.e., $S' = \emptyset$ and sNUB holds trivially. □

Acknowledgments

We thank Noam Nisan and Tomer Ezra for insightful comments.

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No. 866132), and by the Israel Science Foundation (grant number 317/17).

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